Mathematical Reviews

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Mathematical Reviews

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HISTORY

Miller, G. A. A fourth lesson in the history of mathematics. Nat. Math. Mag. **17**, 13–20 (1942). [MF 7236]

von Erhardt, Rudolf and von Erhardt-Siebold, Erika. The helix in Plato's astronomy. Isis **34**, 108–110 (1942). [MF 7415]

Discussion of a difficult passage in Plato's Timaeus, concerning the movement of the planets. *O. Neugebauer.*

Neugebauer, O. On two astronomical passages in Plutarch's De Animae Procreatione in Timaeo. Amer. J. Philology **58**, 455–459 (1942). [MF 7136]

The passages in question are interpreted as dealing with the unequal lengths of the seasons and with the change in the length of the days. *O. Schmidt* (Providence, R. I.).

Romañá, Antonio. The astronomical work of Galileo Galilei. Revista Mat. Hisp.-Amer. (4) **2**, 125–178 (1942). (Spanish) [MF 7323]

Levi, B. The postulate of Archimedes. From Euclid to Galileo: modern concepts. Math. Notae **2**, 109–141 (1942). (Spanish) [MF 7795]

Chapman, S. Blaise Pascal (1623–1662). Tercentenary of the calculating machine. Nature **150**, 508–509 (1942). [MF 7461]

Bell, E. T. Newton after three centuries. Amer. Math. Monthly **49**, 553–575 (1942). [MF 7483]

Santaló, Luis A. Isaac Newton and the binomial theorem. Math. Notae **2**, 61–72 (1942). (Spanish) [MF 7252]

Ferguson, Allan. Newton and the "Principia." Philos. Mag. (7) **33**, 871–888 (1942). [MF 7788]

von Kármán, Th. Isaac Newton and aerodynamics. J. Aeronaut. Sci. **9**, 521–522, 548 (1942). [MF 7556]

✶Sergescu, Petre. An episode in the struggle for the triumph of differential calculus; the Rolle-Sourin polemic 1702–1705. Studies dedicated to the memory of the great Nicolas Iorga, 17 pp. Bucharest, 1942. (Rumanian)

Sergescu, P. Sur l'identité des auteurs de quelques articles mathématiques, insérés dans "Le Journal des Savants" 1684–1703. An. Acad. Romane. Mem. Secţ. Ştiinţifice (3) **17**, no. 9, 21 pp. (1942). [MF 7224]

Rolle, de l'Hospital, Leibniz and others are the authors in question. *O. Neugebauer* (Providence, R. I.).

Loria, Gino. La "courbe catoptrique" d'Euler. Enseignement Math. **38**, 250–275 (1942). [MF 7302]

Richeson, A. W. Laplace's contribution to pure mathematics. Nat. Math. Mag. **17**, 73–78 (1942). [MF 7467]

Loria, Gino. Perfectionnements, évolution, métamorphoses du concept de "coordonnées." Contribution à l'histoire de la géométrie analytique. Mathematica, Timişoara **18**, 125–145 (1942). [MF 7427]

Delevsky, Jacques. L'invention de la projection de Mercator et les enseignements de son histoire. Isis **34**, 110–117 (1942). [MF 7416]

Whittaker, E. T. Aristotle, Newton, Einstein. Proc. Roy. Soc. Edinburgh, Sect. A. **61**, 231–246 (1942). [MF 7389]

Brasch, Frederick E. James Logan, a colonial mathematical scholar, and the first copy of Newton's Principia to arrive in the colony. Proc. Amer. Philos. Soc. **86**, 3–12 (1942). [MF 7264]

✶Fueter, Eduard. Geschichte der exakten Wissenschaften in der Schweizerischen Aufklärung (1680–1780). Sauerländer et Cie., Aarau and Leipzig, 1941. 336 pp.

Zchakaja, D. Über die mathematischen Kenntnisse in Georgien im XVIII Jahrhundert. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **9**, 207–215 (1941). (Russian. German summary) [MF 7382]

Garcia, Godofredo. The scientific work of Professor George D. Birkoff. Revista Ci., Lima **44**, 187–232 (1942). (Spanish) [MF 7115]

Biography: S. A. Chaplygin's fifty years of outstanding work as research scientist. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 131–148 (1941). (Russian. English summary) (1 plate) [MF 7703]

Obituary: Andrew Russell Forsyth. 1858–1942. Obit. Notices Roy. Soc. London **4**, 209–227 (1942). [MF 7554]

Obituary: Andrew Russell Forsyth. 1858–1942. Math. Gaz. **26**, 117–118 (1942). [MF 7103]

Biography: On the seventieth anniversary of the birth of B. G. Galerkin. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 331–334 (1941). (Russian. English summary) (1 plate) [MF 7718]

Fehr, H. Obituary: Henri Lebesgue. 1875–1941. Enseignement Math. **38**, 330–332 (1942). [MF 7308]

Rosenblatt, Alfred. Obituary: Henri Lebesgue. Revista Ci., Lima **44**, 357–364 (1942). (Spanish) (1 plate) [MF 7542]

Sergescu, Petre. Life and mathematical work of Henri Lebesgue. Monografii Mat. 7, 15–23 (1942). (Rumanian) [MF 7223]

Stoilow, S. Mathematical work of Henri Lebesgue. Mathematica, Timişoara 18, 13–25 (1942). (Rumanian) [MF 7212]

Buhl, A. Obituary: Tullio Levi-Civita. 1873–1941. Enseignement Math. 38, 350–351 (1942). [MF 7311]

Levi, B. Obituary: Tullio Levi-Civita (1873–1941). Math. Notae 2, 155–159 (1942). (Spanish) [MF 7796]

Biography: The sixtieth anniversary of the birth of Professor Doctor E. L. Nicolai. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] 5, 3–10 (1941). (Russian. English summary) (1 plate) [MF 7698]

Angheluţă, Th. Life and mathematical work of Emile Picard. Monografii Mat. 7, 1–12 (1942). (Rumanian) [MF 7222]

Buhl, A. Obituary: Emile Picard. 1856–1941. Enseignement Math. 38, 348–350 (1942). [MF 7310]

Rosenblatt, Alfred. Obituary: Émile Picard. Revista Ci., Lima 44, 311–356 (1942). (Spanish) (1 plate) [MF 7541]

Wavre, R. Obituary: Vito Volterra. 1860–1940. Enseignement Math. 38, 347–348 (1942). [MF 7309]

ALGEBRA

✶Levi, F. W. Algebra. Vol. 1. University of Calcutta, Calcutta, 1942. xii+305 pp.

The book covers part of a post-graduate pure mathematics course. It had first been published in form of lecture notes. An outline of the contents is as follows: Introductory remarks, mathematical induction, permutations. I. Systems of linear equations, vector spaces, matrices and linear transformations, determinants. II. Fundamentals of general algebra: Modules, rings, and fields. Polynomials, derivatives, factorisation. Existence of roots of algebraic equations in suitable extension fields. Structure of extension fields. III. General algebra, specified theory: Cyclotomic polynomials. Galois fields. The field of complex numbers. Irreducibility of polynomials with rational coefficients. Symmetric polynomials. Solution of cubic and biquadratic equations by radicals. Resultants. Algebraically closed fields. Proof that the field of all complex numbers is algebraically closed. IV. Continued fractions: General properties. Periodic continued fractions. Applications to the theory of numbers. The field of formal power series in $1/x$, expansion of its elements into continued fractions. Questions of convergence. V. Approximation of roots: Horner's scheme. Roots of real polynomials. Budan-Fourier's theorem, Sturm's theorem. Newton's and Graeffe's methods, Poulain's theorem. Roots of complex polynomials, position of the roots of the derivative of a polynomial. VI. Matrices: General properties. Characteristic polynomial, similarity of matrices, the Jordan normal form. Elementary divisors. Unitary, orthogonal, symmetric, and skew symmetric matrices. Bilinear, quadratic, and Hermitian forms.

As will be seen from this description of the contents, an introduction to abstract algebra is given, and the numerical problems are treated. The book is well written, and many examples should further increase its value as a text book.

R. Brauer (Toronto, Ont.).

✶Artin, Emil. Galois Theory. Edited and supplemented with a section on applications by Arthur N. Milgram. Notre Dame Mathematical Lectures, no. 2. University of Notre Dame, Notre Dame, Ind., 1942. i+70 pp. $1.25

The modern simplified treatment of the Galois theory began with the recognition that the allowable permutations of the roots of a polynomial equation, as employed in the older Galois theory, are actually automorphisms $\sigma : a \to \sigma(a)$ of the field E generated by the roots. In this elegant presentation, Artin carries out the implications of this "automorphism" approach, using systematically the fact that each automorphism σ can be regarded as a character (in E) of the multiplicative group of E. Starting from scratch with the definitions of fields and vector spaces, he derives the fundamental theorem of Galois theory in a brief 40 pages. His masterful presentation is to be recommended heartily to all mathematicians.

The treatment starts with any field E and a finite group G of automorphisms of E and constructs the fixed field F of G as the set of all $a \varepsilon E$ with $a = \sigma(a)$ for every automorphism σ of G. Regarding the automorphisms as characters, and using to the full the elementary theorem that m homogeneous linear equations in $n > m$ unknowns always have a nontrivial solution, Artin proves very quickly that the degree of E over the fixed field F equals the order of G. The fundamental theorem follows at once. Applications include finite fields, Kummer fields, "natural" irrationalities and a proof of Emmy Noether's ("principal genus") theorem, that the only solutions $x_\sigma \varepsilon E$ of the equations $x_\sigma \cdot \sigma(x_\tau) = x_{\sigma\tau}$ are $x_\sigma = a/\sigma(a)$. There is an elegant proof that a finite extension E of F is simple if and only if there are but a finite number of intermediate fields. The section by Milgram treats the solution of equations by radicals, the general equation of degree n, solvable equations of prime degree and constructions by ruler and compass. Misprints: in Corollary 2, p. 33 read "finite groups"; in the fundamental theorem, p. 36, add the hypothesis that the irreducible factors of $p(x)$ are separable. Note also that an alternative proof for theorems 10 and 15 is indicated on page 56.

S. MacLane (Cambridge, Mass.).

Poivert, Jules. Étude sur la pseudo-résolvante. Rev. Trimest. Canad. 28, 1–15 (1942). [MF 6376]

This paper may be described as an attempt to investigate the algebraic solvability of equations $x^n + a_2x^{n-2} + \cdots + a_n = 0$ without the systematic use of group theory. Use is made of the Lagrange-Abel formula $x = R_1^{1/n} + R_2^{1/n} + \cdots + R_{n-1}^{1/n}$ for the roots. For the cubic $x^3 + px + q = 0$, we have $x_1 + x_2 = x$, $x_1x_2 = -p/3$, $x_1^3 + x_2^3 = q$. Considering this as a system of three equations for x_1, x_2, we may either solve the first two equations and substitute into the third, thus obtaining the original equation, or we may solve the second and third equations and substitute into the first, obtaining the solution x. For the general equation $x^n + a_2x^{n-2} + \cdots + a_n = 0$ we start out from a corresponding system of

equations: $\sum x_1=x$, $\sum x_1x_2=b_2$, $\sum x_1x_2x_3=b_3$, $\cdots$, $x_1x_2\cdots x_{n-1}=b_{n-1}$, together with $x_1^n+x_2^n+\cdots+x_{n-1}^n=c$. The $x_1, \cdots, x_{n-1}$ are identified with the $R_1, \cdots, R_{n-1}$ above; the $b_2, \cdots, b_{n-1}$, c can be computed. The present paper deals with $n=5$. The fourth degree equation with roots $x_1, \cdots, x_4$ (that is, $R_1^{1/5}, \cdots$) is a resolvent of the given equation. By eliminating $x_1, \cdots, x_4$ in the two ways, as above, it is shown that the coefficients of this resolvent are themselves roots of fifth degree equations, and that therefore the algebraic solution cannot be obtained in this manner. The method permits the establishment of classes of quintics which can be solved algebraically, and gives the solutions. *A. J. Kempner.*

Bhalotra, Yashpaulraj. A criterion for the solubility by radicals of the general quintic. Math. Student 9, 161–163 (1941). [MF 7033]

Extension of the following result from Cajori's "Introduction to the Modern Theory of Equations" [Macmillan Co., New York, 1904, pp. 229–230]: Necessary and sufficient condition for the irrational quintic $x^5+cx+d=0$ to be solvable by radicals is that

$$(z^3-5cz^2+15c^2z+5c^3)^2=(4^4c^5+5^5d^4)z$$

have a rational root. The author applies Cajori's method to establish this generalization: The irrational quintic $x^5+ax^3+bx^2+cx+d=0$ is solvable by radicals if and only if

$$\begin{aligned}\{z^3-(20c+3a^2)z^2+(240c^2-400bd-8a^2c+16ab^2+3a^4)z\\ +(320c^3-1600bcd-64b^4+4000ad^2+224ab^2c-176a^2c^2\\ -80a^2bd-16a^3b^2+28a^4c-a^6)\}^2=2^{10}D^2z,\end{aligned}$$

where D is the discriminant of the quintic, has a rational root. Although it is not explicitly stated, the coefficients a, b, c, d seem to be assumed in the natural domain, etc. As an application two quintics [Ramanujan], $x^5-x^4+x^3-2x^2+3x-1=0$ and $x^5+2x^4+2x^3+x^2-1=0$, are shown to be solvable by radicals. *A. J. Kempner.*

Lubelski, S. Über zwei Wegnersche Sätze. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 395–398 (1941). (Russian. German summary) [MF 6827]

The author gives two theorems having to do with the reducibility of a polynomial and its decomposition into factors with respect to a prime modulus. The first theorem is as follows. Let m be a positive integer and q any prime not dividing the totient $\phi(m)$ of m. Let K be the field defined by the irreducible equation $x^n+a_1x^{n-1}+\cdots=0$, where $n=qd$ and the a's are integers. Then there exist infinitely many primes $p\equiv 1 \pmod m$ which have prime ideal factors in K of degree q. The second theorem provides a generalization of the counterexample $(x^2+x+1)(x^3-A)$ [given by van der Waerden, Math. Ann. 109, 679–680 (1933)] of a polynomial without linear factors (A is not a perfect cube) which nevertheless has a linear factor modulo p for all primes p. The class of all polynomials of prime degree is enlarged to include any irreducible polynomial of degree q^α whose field contains a subfield of degree $q^{\alpha-1}$ which in turn has a subfield of degree $q^{\alpha-2}$, and so on, until a subfield of prime degree is reached; such polynomials are called regular. The second theorem states that, if $f(x)=x^n+a_1x^{n-1}+\cdots$ has integer coefficients and a nonsquare discriminant D and is regular, then the polynomial $(x^2-D)f(x)$ which has no linear factor nevertheless has p linear factors modulo p for almost all primes p. *D. H. Lehmer.*

Tschebotaröw, N. On the methods of Sturm and Fourier for transcendent functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 34, 2–4 (1942). [MF 7451]

The classical theorem of Sturm which correlates the number of roots of an algebraic equation in an interval with the losses in sign variation in that interval suffered by a certain sequence of functions is extended to transcendental equations of the form $P(x, \cos x, \sin x)=0$, where P is a real polynomial. The Sturm's sequence is formed by differentiating P totally with respect to x and then performing on the resulting functions P and P' the usual modified algorithm for finding the greatest common divisor, regarding $\cos x$ and $\sin x$ as constants. As applications, bounds are obtained beyond which every interval between successive integral multiples of 2π contains the same number of roots, also results on the interlacing of roots of two such equations. A similar extension of the Fourier method is indicated.
P. W. Ketchum (Urbana, Ill.).

Petiau, Gérard. Sur un système de nombres hypercomplexes dérivés des nombres de Clifford. Mathematica, Timişoara 18, 77–96 (1942). [MF 7424]

Formal properties are derived for systems built up from the direct products of Clifford numbers; that is, elements of the algebra of order 2^n consisting of linear combinations with complex coefficients of the elements $e_0=1, \gamma_1, \gamma_2, \cdots, \gamma_n$ satisfying $\gamma_i\gamma_j+\gamma_j\gamma_i=2\delta_{ij}\cdot 1$ and their products.
A. H. Taub (Princeton, N. J.).

Niven, Ivan. The roots of a quaternion. Amer. Math. Monthly 49, 386–388 (1942). [MF 6793]

The author investigates the number of solutions of the quaternion equation $\xi^m=\alpha$; the existence of such solutions is already known for any such algebraic equation [Amer. Math. Monthly 48, 654–661(1941); these Rev. 3, 264]. The result is that there are exactly m distinct mth roots of α if α is a quaternion that is not a real number; if α is real there are infinitely many mth roots unless $m=2$ and α is positive, in which case there are just the two square roots $\pm\sqrt{\alpha}$. The proof, which is very simple, gives an explicit determination of the roots in question.
H. W. Brinkmann (Swarthmore, Pa.).

Brand, Louis. The roots of a quaternion. Amer. Math. Monthly 49, 519–520 (1942). [MF 7342]

The author derives Niven's results on the roots of a quaternion [cf. the preceding review] by a simple method which is entirely similar to finding the roots of a complex number. *H. W. Brinkmann* (Swarthmore, Pa.).

Balachandran, K. Determinants connected with homogeneous products and symmetric functions of the roots of an equation. Math. Student 9, 137–142 (1941). [MF 7029]

Let S_r denote the rth elementary symmetric function of n indeterminates and Σ_r the sum of the rth powers of these indeterminates. The first r of Newton's identities are solved by means of determinants for S_r, $S_r=\Delta(n, r)/r!$, where $\Delta(n, r)$ is an r-rowed determinant. Some simple properties of $\Delta(n, r)$ are discussed as well as those of an analogous determinant, which expresses the value of H_r (the sum of all homogeneous products of degree r with repetitions allowed) in terms of the Σ_i. *J. Williamson.*

Turnbull, H. W. On certain modular determinants. Edinburgh Math. Notes no. **32**, 23–30 (1941). [MF 7551]

The author defines, for any odd prime p, the determinant $\Delta=|a_{rs}|_n$, where a_{rs} is the smallest positive integer such that $ra_{rs}\equiv s \pmod p$, and $n=\frac{1}{2}(p-1)$. He shows that $\Delta=q(-p)^{n-1}$, where q is an integer which arises in the form of a determinant of 0's and 1's. E. Malo [L'Intermédiaire des Math. **21**, 173–176 (1914)] had suspected that $q=1$, as is the case for every prime $p<29$, but in fact $q=8$ when $p=29$, and $q=9$ when $p=31$.

The latter half of the paper is concerned with a modification of this determinant, a_{rs} being diminished by p whenever it is greater than $\frac{1}{2}p$. Then, apart from sign, each row and each column contains some permutation of the first n integers. The value of the determinant is of the same general form as before, with possibly a different q. By suitable changes of sign and rearrangement of rows and columns, the determinant can be put into the "persymmetric" form $|c_{rs}|_n$, where $c_{rs}\equiv a^{r+s-2} \pmod p$, $-n<c_{rs}\leqq n$. Here a is a quadratic residue or non-residue according as n is odd or even. In the former case the determinant is a Latin square: each of the first n integers occurs in every row and every column, the negative sign always accompanying the same integer. *H. S. M. Coxeter* (Toronto, Ont.).

Dresden, Arnold. On the iteration of linear homogeneous transformations. Bull. Amer. Math. Soc. **48**, 577–579 (1942). [MF 7052]

Dresden, Arnold. A correction to "On the iteration of linear homogeneous transformations." Bull. Amer. Math. Soc. **48**, 949 (1942). [MF 7519]

The author proves that the iteration of the linear homogeneous transformation $x_k'=\sum_{j=1}^{n}a_{kj}x_j$, $(k=1,\cdots,n)$, will converge for every initial set x_k if and only if all the roots of the characteristic equation of the matrix (a_{ij}) are less than unity in absolute value, except that the roots which are involved only in linear elementary divisors may be equal to unity. In the "correction" the author acknowledges that the result of this paper has been previously established by Oldenburger [Duke Math. J. **6**, 357–361 (1940); these Rev. **1**, 324]. *W. T. Reid* (Chicago, Ill.).

Foulkes, H. O. Collineatory transformation of a square matrix into its transpose. J. London Math. Soc. **17**, 70–80 (1942). [MF 7314]

This paper is a study of the equation $HA=A'H$, where A is a given square matrix of order n and H an unknown matrix, also of order n. Explicit solutions are given for $n=2, 3$ and also methods of obtaining solutions for higher values of n. Various related questions are discussed; in particular, the author indicates a method of obtaining matrices commutative with A. *N. H. McCoy.*

Williamson, John. A generalization of the polar representation of nonsingular matrices. Bull. Amer. Math. Soc. **48**, 856–863 (1942). [MF 7500]

Let H be any nonsingular Hermitian matrix with complex elements, and let A be nonsingular. $A=DR$ is called an H-representation of A if $DH=HD^*$ and $RHR^*=H$. If in particular H is the identity matrix, then R is unitary and D is positive definite Hermitian, so that the H-representation becomes the ordinary polar representation. It is shown that A has an H-representation if and only if the negative elementary divisors of AHA^*-xH occur in pairs and exactly half of the indices associated with each negative elementary divisor are positive; D is unique if and only if every elementary divisor is positive and of odd order. More generally, if H is nonsingular Hermitian and A is nonsingular, there exists a matrix V whose square is the identity such that $A=DR$, $RHR^*=VH$ and DH is Hermitian with the same signature as H. *C. C. MacDuffee* (New York).

Smiley, M. F. The rational canonical form of a matrix. Amer. Math. Monthly **49**, 451–454 (1942). [MF 7148]

An elementary similarity transformation on a matrix A is accomplished by applying an elementary transformation to the rows of A and the inverse transformation to the columns of A. If $B=PAP^{-1}$, then B can be obtained from A by a succession of elementary similarity transformations, and conversely. The author describes a method for reducing, by means of elementary similarity transformations, a matrix A with elements in any field to a direct sum of canonical blocks, that is, submatrices with 1's just above the main diagonal and the negatives of the coefficients of the characteristic equation in the last row and 0's elsewhere. The rational canonical form of A may then be computed by the same method, or more simply by invariant factor theory. *C. C. MacDuffee* (New York, N. Y.).

Etherington, I. M. H. Some problems of non-associative combinations. I. Edinburgh Math. Notes no. **32**, 1–6 (1941). [MF 7545]

A connection is disclosed between the partition of a convex polygon by nonintersecting diagonals and the insertion of brackets in a product. The problem is practically equivalent to the construction of Cayley's trees. The enumeration of these partitions leads to a generating function $y=f(x)$ which satisfies an algebraic equation

$$y=x+y^a+y^b+\cdots \qquad a, b, \cdots>1.$$

In simple cases the solution of the equation is found as a power series in x, the coefficient A_n of x^n giving the number of partitions of an $(n+1)$-gon. The known formula in terms of factorials is obtained for the number of partitions into triangles, that is, for the number of associations of a product of n factors. In the more complicated cases A_n may be obtained by successive approximations. *C. C. MacDuffee* (New York, N. Y.).

Erdélyi, A. and Etherington, I. M. H. Some problems of non-associative combinations. II. Edinburgh Math. Notes no. **32**, 7–12 (1941). [MF 7546]

The writers apply a method of Birkeland [C. R. Acad. Sci. Paris **171** (1920) and **172** (1921)] using contour integrals to enumerate the partitions of a polygon described in the preceding review. More generally, the number of partitions of an $(n+1)$-gon into α $(a+1)$-gons, β $(b+1)$-gons, $\cdots$, where n, α, a, β, b, $\cdots$ are all given numbers such that $n-1=\alpha(a-1)+\beta(b-1)+\cdots$, is shown to be

$$A_{n\alpha\beta}=\frac{(n-1+\alpha+\beta+\cdots)!}{n!\,\alpha!\,\beta!\cdots}$$

and hence the value of the A_n of the preceding review is

$$\sum_{\alpha,\beta,\cdots} A_{n\alpha\beta\cdots} \quad \text{where} \quad \alpha(a-1)+\beta(b-1)+\cdots=n-1.$$

A note added in proof states that similar results were obtained simultaneously by G. Belardinelli [Monatsh. Math. Phys. **48**, 381–388 (1939); these Rev. **1**, 117]. *C. C. MacDuffee* (New York, N. Y.).

Kerawala, S. M. The enumeration of the Latin rectangle of depth three by means of a difference equation. Bull. Calcutta Math. Soc. **33**, 119–127 (1941). [MF 7027]

The problem of the enumeration of the Latin rectangles of depth three is a simple generalization of the problem of "derangements" (problème des rencontres) in which one asks for the number of permutations of the n letters

$$(1) \qquad a_1, a_2, a_3, \cdots, a_n,$$

in which no letter occupies a position whose rank is equal to its subscript. The number $2K_n$ in question was found by Euler to be the product of $n!$ by the first n terms of the series for e^{-1}, a number known as subfactorial n. The problem considered in this paper is that of finding the number of ways a rectangular array of three rows of which the first is (1) may be written using the letters of (1) in such a way that each of the n columns is made up of three distinct letters. The general problem of s rows was considered by MacMahon who gave an "operational solution" [Trans. Cambridge Philos. Soc. **16**, 262–290 (1898)]. The case of $s=3$ was dealt with in detail by Jacob [Proc. London Math. Soc. (2) **31**, 329–354 (1930)] who derived a pair of difference equations for computing the corresponding number ${}_3K_n$, and all but conjectured that ${}_3K_n/(n!)^2$ tends to e^{-3}. The present author points out that Jacob made a slight error in his difference equations so that his tables are incorrect. The number ${}_3K_n$ is now shown to satisfy a single linear difference equation of the fifth order; ${}_3K_n$ is tabulated for $n\leqq 15$ and the above ratio to $n=25$. Finally it is shown that this ratio (for $n\leqq 5$) forms a monotone increasing sequence approaching a limit l such that $.0497865<l<.0497884$. ($e^{-3}=.0497870684\cdots$). *D. H. Lehmer.*

Abstract Algebra

Klein-Barmen, Fritz. Molekulare Verbände. Math. Z. **47**, 373–394 (1941). [MF 6724]

A "molecular" lattice is a lattice whose join-irreducible elements not 0 form chains of "primary" elements over different "prime" elements covering 0. The author obtains a "canonical representation" for each element x of a molecular lattice as a join of maximal primaries contained in x. He proves that a molecular lattice is distributive if and only if every element can be represented uniquely as a join of primaries ("unimolecularity" condition). [Cf. also his paper in Math. Ann. **106**, 114–130 (1932).] He also proves that, if, in a lattice L, it is impossible to find three mutually incomparable elements, then the lattice is distributive. *G. Birkhoff* (Cambridge, Mass.).

Wilcox, L. R. A note on complementation in lattices. Bull. Amer. Math. Soc. **48**, 453–458 (1942). [MF 6713]

The author discusses "right-complemented" and "left-complemented" lattices; the definitions depend on his concept of a modular pair [Ann. of Math. (2) **40**, 490–505 (1939)]. He proves that if a lattice L is left-complemented, then it is right-complemented, and the notions of a right-complement and a left-complement are effectively equivalent. Examples of left-complemented lattices are given, including affine geometries; this is related to ideas of Menger and Saul Gorn. *G. Birkhoff* (Cambridge, Mass.).

Sagastume Berra, A. E. p-adic numbers and topology. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista **2**, 125–145 (1941). (Spanish) [MF 7133]

This expository paper gives a topological development of the p-adic and $\mathfrak{p}$-adic numbers of Hensel, starting with the p-adic topology in the ring of integers of an algebraic number field K, defining the corresponding complete field and proving it totally disconnected and locally compact. *S. MacLane* (Cambridge, Mass.).

Jennings, S. A. Central chains of ideals in an associative ring. Duke Math. J. **9**, 341–355 (1942). [MF 6872]

For ideals A, B of an associative ring R the author defines the "commutator-ideal" $A\circ B$ as the ideal generated by the commutators $ab-ba$. If the chain of ideals $H_1=R$, $H_{i+1}=R\circ H_i$ terminates with 0, R is said to be of finite class. A (necessary and) sufficient condition for R to be of finite class is the existence of a "central chain," that is, a decreasing sequence M_i of ideals which satisfies $R\circ M_i\subseteqq M_{i+1}$ and terminates with zero. It is shown that the products H_pM_σ and $M_\sigma H_p$ are contained in $M_{p+\sigma-1}$; this implies, in particular, that the "derived ring" $R\circ R$ is nilpotent if R is of finite class. The relations $M_i\supseteqq H_i$ justify the name "lower central chain" for H_i. A section of the paper is devoted to "upper central chains" defined by $Z_0=0$, $Z_{i+1}=$largest ideal satisfying $R\circ Z_{i+1}\subseteqq Z_i$. Next, solvability is introduced as follows: Let $R^{(0)}=R$, and $R^{(i+1)}$ the ideal in $R^{(i)}$ generated by the commutators from $R^{(i)}$. If this decreasing sequence of rings terminates with zero, the ring R is said to be solvable. The precise analogue of Lie's theorem: if R is solvable, $R\circ R$ is nilpotent, is obtained without any additional finiteness conditions. Necessary and sufficient for solvability is the existence of a nilpotent ideal N such that R/N is commutative.

The elements of an associative ring, under the multiplication $a\circ b$, constitute the "associated Lie ring" $\mathfrak{R}$ of R. It is not hard to show that, if R is solvable (of finite class), $\mathfrak{R}$ is solvable (nilpotent). The converse problem is more difficult. Here the following far reaching but not quite exhaustive results are obtained: (1) If R is an algebra, not of characteristic 2 (the latter condition is shown to be indispensable), then R is solvable if $\mathfrak{R}$ is; (2) if R is an algebra of any characteristic, it is of finite class if $\mathfrak{R}$ is nilpotent; (3) if R is already known to be solvable, it will be of finite class if $\mathfrak{R}$ is nilpotent. The last section of the paper deals with nilrings which are generated by a finite number of elements. Such a ring is nilpotent once it is solvable. *M. A. Zorn* (Los Angeles, Calif.).

Everett, C. J., Jr. An extension theory for rings. Amer. J. Math. **64**, 363–370 (1942). [MF 6438]

An extension theory for rings is developed here, parallel to the extension theory for groups [H. Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig, 1937, p. 89]. If a ring E contains an ideal N with $E/N=F$, E is called an extension of N by F. For each element σ of F one may select a representative S_σ in E; the left and right multiplications of N by S_σ induce endomorphisms Λ_σ and P_σ of N. Addition and multiplication tables for the representatives give two factor sets, which satisfy suitable associativity conditions. These factor sets, with Λ_σ and P_σ, determine the extension. A change of representatives leads to an "equivalent" factor system. If N is a null ring ($N^2=0$) the classes of equivalent factor sets for fixed Λ_σ, P_σ form a group, analogous to the group of group extensions. The existence of splitting rings for an arbitrary extension E is proved by

a method giving a new proof of the corresponding group-theoretical theorem. The existence of a left holomorph of N, under suitable hypothesis, is established. Finally the extension theory is applied to prove the existence of rings of an arbitrary right and left type, in the sense of a previous paper [Duke Math. J. **5**, 623–627 (1939); these Rev. **1**, 2].

S. MacLane (Cambridge, Mass.).

Baer, Reinhold. Inverses and zero-divisors. Bull. Amer. Math. Soc. **48**, 630–638 (1942). [MF 7060]

We shall say, in this review, that an element z of an associative ring R has the property A_r if $az=0$ implies $a=0$; property B_r if it is a right inverse, that is, if an element w exists such that wz is left unit for z and right unit for R; property C_r if $Rz=R$. Properties A_l, B_l, C_l are defined similarly by inverting the order of factors. The following chain conditions are used by the author: (I_l) the minimal condition for "principal left ideals" of form aR, (II_l) the minimal condition for "zero-dividing left ideals," which are sets defined by an equation $xa=0$, (III_l) the maximum condition for the same. Conditions with subscript r are defined correspondingly, and the existence of a unit element 1 is denoted by (IV). [Cf. the reviewer's thesis, Abh. Math. Sem. Hansischen Univ. **8**, 123–147 (1930), in particular, p. 137.] With these conventions we report the following theorems: (1) condition (I_l) implies that A_r and B_r are equivalent; (2) conditions (I_l) and (I_r) make A_r, A_l, B_r and B_l equivalent; (3) the conditions (I_l) and (II_l) yield equivalence of B_r and C_r; (5) if the conditions (IV), (II_l) or (IV), (III_l) hold, then the equations $uv=1$ and $vu=1$ are equivalent. The latter theorem holds also if u and v are matrices over the ring R. Results for rings with minimum condition for left ideals: (corollary to 6) there exists a left unit for R if every element has a left unit; (7) a unit element exists if (1)$Rx=0$ or $xR=0$ implies $x=0$ and (2) $RP=PR$, where P is the radical of R.

The paper concludes with an observation about the commutative law of addition. Generalizing the fact that this law follows from the others (for a distributive ring) if a unit element is present, it is shown: addition is commutative in RR, and the commutator group C of the addition group satisfies $CR=RC=0$. *M. A. Zorn* (Los Angeles, Calif.).

MacDuffee, C. C. Products and norms of ideals. Amer. J. Math. **64**, 646–652 (1942). [MF 7168]

In this paper the author continues his investigations on ideals in a Frobenius algebra [Monatsh. Math. Phys. **48**, 293–313 (1939); these Rev. **1**, 100]. For each class of left ideals $\mathfrak{a}$ a definite minor class of ideal matrices is selected, and all ideal matrices are taken from these minor classes. Then each ideal $\mathfrak{a}$ determines a unique ideal matrix A. With the ideal $\mathfrak{a}$ there are associated h further matrices A_b, where b ranges over the ideal classes and where h is the class number of the domain. Let $\mathfrak{b}$ be an ideal of the class b which corresponds to the ideal matrix B. Then the product $\mathfrak{a}\cdot\mathfrak{b}$ corresponds to the ideal matrix A_bB. Besides its ordinary norm $|A|$ the ideal $\mathfrak{a}$ has h further norms $|A_b|$. It is shown that each norm of a product is equal to the product of certain properly chosen norms of the factors.

R. Brauer (Toronto, Ont.).

Brauer, Richard. On the nilpotency of the radical of a ring. Bull. Amer. Math. Soc. **48**, 752–758 (1942). [MF 7279]

This paper contains a simple derivation of Hopkins' result [Duke Math. J. **4**, 664–667 (1938)] that the structure theory of noncommutative rings can be based on the assumption of only the minimum conditions for left-ideals. The author also shows that it is sufficient to assume only the minimum condition for sets of such two-sided ideals as contain only nilpotent elements in order to prove the nilpotency of the radical. Assuming that the radical N is nilpotent and that the minimal condition holds for sets of left-ideals containing N, the author proves that every left-ideal of the ring R is a direct sum of primitive left-ideals $R\epsilon_i$ and a nilpotent left-ideal $\mathfrak{n}$, where the ϵ_i are idempotent such that $\epsilon_i\epsilon_j=0$ for $i\neq j$ and $\mathfrak{n}\epsilon_i=0$. Then $\zeta=\epsilon_1+\cdots+\epsilon_n$ is a unit element (mod N), and, if R has a unit element, it is ζ. *C. C. MacDuffee* (New York, N. Y.).

Newman, M. H. A. A characterisation of Boolean lattices and rings. J. London Math. Soc. **16**, 256–272 (1941). [MF 6620]

The author considers certain generalizations of Boolean (lattices) algebras obtained through a weakening of the axioms. Through specialization one obtains certain characterizations of Boolean algebras. The paper is based upon the concept of "double algebras" defined as a system with two operations $a\cup b$ and $a\cdot b$. It is distributive when $a(b\cup c)=ab\cup ac$, $(b\cup c)a=ba\cup ca$, and idempotent when $a\cdot a=a$. An element e is a zero if $e\cup a=a\cup e=a$ for all a and a unit if $ea=ae=a$. Left and right units and zeros are defined analogously. Finally two elements 0 and 1 are called an extreme pair if for every a the equations $x\cup a=1$, $xa=0$ have some solution.

The main theorem is then that in order that the double algebra B be the direct union of a nonassociative Boolean ring with unit and a Boolean lattice it is necessary and sufficient that B be distributive and idempotent and that the 0 of at least one extreme pair be a left zero. The condition may also be stated in various other forms, for instance, B shall be distributive and in at least one extreme pair 0 be a left zero and 1 a left unit. The paper also contains an extended analysis of the interrelations between various axiomatic systems. *O. Ore* (New Haven, Conn.).

Newman, M. H. A. Relatively complemented algebras. J. London Math. Soc. 17, 34–47 (1942). [MF 6973]

In this paper the results previously obtained for complemented algebras are extended to relatively complemented algebras. A double algebra is defined as in the preceding review. An element x is a b-complement of a (relative to 0) when $x\cup a=b$, $x\cdot a=0$. An element 0 is said to be a right ω if a b-complement of a relative to 0 exists whenever $ab=a$. An element b is called a majorant of a when $ab=a$. An element a is odd when $a\cup a=a$, even when $a\cup a=0$, where 0 is a left zero. For an algebra A which is distributive, idempotent and containing at least one right ω one can prove that it is the direct union of its odd and even subalgebras.

To derive a result containing as a special case the decomposition theorem previously obtained for complemented algebras it is necessary to make further assumptions. A generalized Boolean lattice is defined as a distributive lattice with a right ω. A nonassociative Boolean ring is a distributive, idempotent even algebra in which there is a right zero and the union operation $a\cup b$ is associative. It becomes a Boolean ring in the sense of Stone when it is associative with respect to the product $a\cdot b$. A mixed nonassociative algebra is the direct union $A_1\cup A_2$ of an nonassociative Boolean ring and a generalized Boolean lattice and a mixed algebra is the direct union of a Boolean ring and a generalized Boolean lattice. To state the main results,

let A be a double algebra which is distributive, idempotent and contains a right ω. Then: (1) A is a mixed nonassociative algebra if every three elements have a common majorant. (2) A is a generalized Boolean lattice if it is odd and every two elements have a common majorant. (3) A is a mixed algebra if it contains a left ω and satisfies the weak associative law $a(bb)=(ab)b$. To conclude, there are certain independence investigations for the various axiomatic conditions. *O. Ore* (New Haven, Conn.).

Dunford, Nelson and Stone, M. H. On the representation theorem for Boolean algebras. Revista Ci., Lima **43**, 447–453 (1941). [MF 6225]

Two proofs of the representation theorem for Boolean algebra. The second one is based on a construction which makes use only of the multiplicative properties of the Boolean algebra. It leads thus to the more general statement: Let M be a system with a binary, idempotent, commutative and associative operation of multiplication. Then M has an isomorphic point set representation which reduces to the usual one in case of distributive lattices, Boolean algebras and Boolean rings. In the case of arbitrary lattices the representation preserves distributive sums. The discussion can be extended to the case of arbitrary partially ordered sets. The paper contains many typographical errors. *F. Bohnenblust* (Princeton, N. J.).

Mercier, André. Beziehungen zwischen den Clifford'schen Zahlen und den Spinoren. Helvetica Phys. Acta **14**, 565–573 (1941).

It is pointed out that both in the four component and two component cases simple (single index) spinors may be considered as ideals in Clifford algebra (algebra of two index spinors). *A. H. Taub* (Princeton, N. J.).

Bruck, Richard H. Generalized Fischer groups and algebras. Bull. Amer. Math. Soc. **48**, 618–626 (1942). [MF 7058]

If K is any field, φ any fixed automorphism of K and A any matrix with elements in K, then A^{φ} denotes the matrix obtained from A by applying the automorphism φ to each of its elements; also $A^{*}=A'^{\varphi}$. A set of n-rowed square matrices which contains A^{*} if it contains A is defined to be a generalized Fischer set (g.F. set). After giving some examples of such sets, the author defines the scalar product

$$F\circ G=\sum c^{\theta}_{i_1\cdots i_r}d_{i_1\cdots i_r},\quad \theta=\varphi^{-1},$$

of the two n^r dimensional vectors F^{θ}, G whose coordinates are, respectively, $c^{\theta}_{i_1\cdots i_r}$ and $d_{i_1\cdots i_r}$. From the fundamental relation $F_{(A^*)}\circ G=F\circ G_{(A)}$, he is able to prove that an irreducible module in a representation of a g.F. set is either nonsingular or of rank zero, and in the latter case its reduced form is given. If, however, the automorphism is involutory, as will be the case in particular if K is the field of real or complex numbers, then each Kronecker product representation of a g.F. set may be written in the form

$$\begin{pmatrix}\mathfrak{A} & & \\ & \mathfrak{B} & \\ * & & \mathfrak{C}\end{pmatrix},$$

where $\mathfrak{B}$ is completely reduced and $\mathfrak{C}$ is related to $\mathfrak{A}$ via the operation (*). *G. de B. Robinson.*

Perlis, Sam. Normal bases of cyclic fields of prime-power degree. Duke Math. J. **9**, 507–517 (1942). [MF 7333]

Suppose that Z is a cyclic extension of prime power degree p^{ϵ} over F. The author discusses necessary and sufficient conditions for an element u in Z to determine with its conjugates a normal basis of Z/F. In case p is equal to the characteristic of F, the trace of u must be different from zero. In the nonmodular case the resulting conditions are more complicated; they involve realizations of subfields of Z by rather intricate linear sets. *O. F. G. Schilling.*

Ancochea, German. Sur quelques théorèmes de la théorie algébrique des corps. Portugaliae Math. **3**, 115–119 (1942). [MF 7085]

Let L/K be a finite separable algebraic extension. The authors give a proof involving only elements of L (and not their conjugates) for the theorem of Abel (existence of a primitive element) and the theorem on the discriminant of a base. *C. Chevalley* (Princeton, N. J.).

Chevalley, Claude. On the composition of fields. Bull. Amer. Math. Soc. **48**, 482–487 (1942). [MF 6717]

For K/k and K'/k arbitrary extensions of the field k the author defines the composites $(\mathfrak{K}/k, \tau, \tau')$ as a field $\mathfrak{K}/k$ together with isomorphisms τ and τ' of K and K', respectively, into subfields K^{τ}, $K'^{\tau'}$ of $\mathfrak{K}$ such that (1) $\mathfrak{K}$ is generated by K^{τ} and $K'^{\tau'}$ and (2) if A and A' are algebraically independent subsets of K and K', respectively, then the set $A^{\tau}\cup A'^{\tau'}$ is algebraically independent in $\mathfrak{K}/k$. Two composites $(\mathfrak{K}_1/k, \tau_1, \tau_1')$ and $(\mathfrak{K}_2/k, \tau_2, \tau_2')$ are equivalent if the isomorphisms $\tau_1^{-1}\tau_2$, $(\tau_1')^{-1}\tau_2'$ between K^{τ_1} and K^{τ_2} and between $K'^{\tau_1'}$ and $K'^{\tau_2'}$ may be extended to an isomorphism between $\mathfrak{K}_1$ and $\mathfrak{K}_2$. The following are the principal results: Any two extensions K/k and K'/k possess at least one composite. Define k to be quasi-algebraically closed (q.a.c.) in K if every element of K which is algebraic over k is purely inseparable over k. Let L and L' be fields such that (1) $k\subset L\subset K$, $k\subset L'\subset K'$; (2) L, L' are algebraic over k; (3) L is q.a.c. in K and L' is q.a.c. in K'. Then the type of composite extension $(K^{\tau}K'^{\tau'}/k, \tau, \tau')$ is determined by the type of composite extension $(L^{\tau}L'^{\tau'}/k, \tau, \tau')$. In particular, if k is q.a.c. in K then there is only one type of composite extension of K and K'. The field $L^{\tau}L'^{\tau'}$ is q.a.c. in $K^{\tau}K'^{\tau'}$. *N. Jacobson* (Chapel Hill, N. C.).

Kaplansky, Irving and Schilling, O. F. G. Some remarks on relatively complete fields. Bull. Amer. Math. Soc. **48**, 744–747 (1942). [MF 7277]

A field K with a rank one valuation V is relatively complete if the Hensel-Rychlik reducibility theorem holds in K, that is, if a polynomial with integral coefficients which has a factorization into relatively prime polynomials (mod P) also has a factorization in K. Generalizing results of F. K. Schmidt [Math. Ann. **108**, 1–25 (1933)] they show that a field K relatively complete with respect to V and complete with respect to another inequivalent valuation is necessarily a multiply complete field. Furthermore, if K has exactly one inseparable extension of every degree and is not relatively complete in any valuation, no finite extension of K is relatively complete. This theorem was suggested by an error in the proof of a subordinate lemma in a previous paper by one of the authors [Lemma 1, Schilling, Amer. J. Math. **62**, 346–352 (1940); these Rev. **1**, 328]. *S. MacLane.*

Hochschild, G. Semi-simple algebras and generalized derivations. Amer. J. Math. **64**, 677–694 (1942). [MF 7171]

As a generalization of the ordinary definition of a derivation in a (not necessarily associative) algebra $\mathfrak{A}$, the author defines a derivation of a subalgebra $\mathfrak{A}$ into an algebra $\mathfrak{B}$ as

a linear mapping $a \to D(a)$ or $\mathfrak{A}$ into $\mathfrak{B}$ satisfying the condition $D(ab)=D(a)b+aD(b)$. The present paper treats derivations of Lie and of associative algebras over a field of characteristic 0. Such an algebra $\mathfrak{A}$ is called reflexive if every derivation of it into any over-algebra $\mathfrak{B}$ of the same kind (Lie or associative) may be extended to an inner derivation $a \to [a, d]$ of $\mathfrak{B}$. One of the main results of the paper is the theorem that a Lie or associative algebra of characteristic 0 is reflexive if and only if it is semi-simple. The proof of this theorem for Lie algebras makes use of a lemma used by Whitehead [Quart. J. Math., Oxford Ser. **8**, 220–237 (1937)] to prove the complete reducibility of the representations of a semi-simple Lie algebra of characteristic 0. An analogue for associative algebras of this lemma is given and this is used as in the Lie case to give a new proof of the complete reducibility of the representations of an associative algebra of characteristic 0. In the special case of derivations of an algebra into itself, it is known that the set of derivations forms a Lie algebra [cf. Jacobson, Trans. Amer. Math. Soc. **42**, 206–224 (1937)]. The author proves that the radical of a Lie (associative) algebra $\mathfrak{A}$ of characteristic 0 is characteristic in the sense that it is mapped into itself by every derivation of $\mathfrak{A}$ [cf. Zassenhaus, Abh. Math. Sem. Hansischen Univ. **13**, 1–100 (1939), in particular, p. 79]. This is used to prove that $\mathfrak{A}$ is semi-simple if and only if its derivation algebra $\mathfrak{D}(\mathfrak{A})$ is semi-simple. *N. Jacobson.*

Whaples, George. Non-analytic class field theory and Grünwald's theorem. Duke Math. J. **9**, 455–473 (1942). [MF 7330]

The contents of this paper may be divided in four parts: (1) An exposition of class-field theory "from the index theorems on"; the object seems to be to treat class field theory from a resolutely finitist point of view. The author makes no use of the notion of group character, even for finite groups; it is not clear to the reviewer why this convenient tool should not be used. A slight mistake occurs in the proof of Theorem 1, when it is stated that "Since the intersections by pairs of K, K' and K'' are all equal to k, the Galois group of $KK'K''/k$ is the direct product of the groups of the individual fields over k". (2) An elegant proof of the ramification theorem is given. The proof is based on a new lemma [Lemma 12] which is also the main instrument in the proof of Grünwald's theorem, and which is quite interesting in itself. (3) Local class field theory is constructed on the base of the theory in the large; the transition is made by means of Lemma 7 of the paper, which is a weaker form of a theorem previously proved by F. K. Schmidt [unpublished]. (4) A very simple proof is given for Grünwald's theorem on the existence of cyclic extensions of a field with certain given local properties. Lemma 13, which is used in the proof, is a purely group-theoretic lemma which is of interest in itself. Some typographical errors are to be guarded against in reading the proof of the fundamental Lemma 12. *C. Chevalley* (Princeton, N. J.).

Kolchin, E. R. Extensions of differential fields. I. Ann. of Math. (2) **43**, 724–729 (1942). [MF 7402]

Let $\mathfrak{F}$ be a differential field of characteristic zero admitting m types of differentiation, for instance, a field of meromorphic functions of m independent variables. Let $\mathfrak{F}$ contain a set of m elements whose Jacobian does not vanish. The paper studies the effect on $\mathfrak{F}$ of the adjunction of an element which satisfies an algebraic partial differential equation with coefficients in $\mathfrak{F}$. It is proved that the adjunction of any finite number of such elements is equivalent to the adjunction of some single element.

J. F. Ritt (New York, N. Y.).

Riblet, Henry J. Factorization of differential ideals. Bull. Amer. Math. Soc. **48**, 575–577 (1942). [MF 7051]

An examination is made of differential rings whose differential ideals satisfy a complicated set of conditions, among which an ascending chain condition figures. The chief result is to the effect that every differential ideal is a product of prime differential ideals. *J. F. Ritt.*

Rathnam, P. Some theorems of algebraic function fields of one variable. Bull. Calcutta Math. Soc. **34**, 33–36 (1942). [MF 7165]

This paper contains some generalizations of classical results to fields of algebraic functions of one variable over an arbitrary coefficient field; for example, the canonical series is free of fixed points. *O. F. G. Schilling.*

ANALYSIS

Ozorio de Almeida, Miguel. On the maxima and minima of certain symmetric functions. Anais Acad. Brasil. Ci. **14**, 99–102 (1942). (Portuguese) [MF 7261]

The functions considered are $\sum_{i=1}^{k} z_i$ and $\sum_{i=1}^{k} z_i^2$, where $z_i=+(a-b(c-x_i)^2)^{\frac{1}{2}}$, $x_0=c-(a/b)^{\frac{1}{2}}<x_i<c+(a/b)^{\frac{1}{2}}=x_{k+1}$, x_i-x_{i-1} is constant for $i=2, \cdots, k$ and $x_1-x_0 \leqq x_i-x_{i-1}$, $x_{k+1}-x_k \leqq x_i-x_{i-1}$. The author attempts to show that each of $\sum z_i$ and $\sum z_i^2$ has a maximum for some distribution of the x_i and that this distribution is the same for either function. The proof is extended to more general types of symmetric functions. The proofs seem questionable, but, as the author states, he is not writing for mathematicians. The problem arose in connection with radiology. *J. V. Wehausen.*

Mandelbrojt, S. and Ulrich, F. E. On a generalization of the problem of quasi-analyticity. Trans. Amer. Math. Soc. **52**, 265–282 (1942). [MF 7121]

The problem of this paper was briefly mentioned on page 118 of Mandelbrojt's recent book [Rice Inst. Pamphlet **29**, no. 1 (1942); cf. these Rev. **3**, 292, also for the basic concepts and notation to be used]. Let $C\{M_n\}$ be a class of infinitely differentiable functions $f(x)$ in the range $0 \leqq x < \infty$. Let $\{\nu_n\}$ $(n=0, 1, \cdots)$ be an increasing sequence of integers $\geqq 0$. We assume throughout that (1) $G=\lim \nu_n/n<2$ and that $f(x)$ is an element of $C\{M_n\}$ satisfying the relations (2) $f^{(\nu_n)}(0)=0$ $(n=0, 1, \cdots)$. If $C\{M_n\}$ is quasi-analytic then $f^{(n)}(0)=0$ $(n=0, 1, \cdots)$ implies $f(x)\equiv 0$. Under what additional conditions will the weaker requirements (2) imply that $f(x)\equiv 0$? A first answer is given by the following theorem 1. If (3) $\int_0^\infty |f(x)|dx<\infty$, (4) $\overline{\lim}\ |f^{(n)}(0)|^{1/n}<\infty$ and the class $C\{M_n\}$ is quasi-analytic, then $f(x)\equiv 0$. The example of $f(x)=x$, belonging to the analytic class $C\{n!\}$, shows that condition (3) can not be dispensed with. Likewise $f(x)=(\sin x/x)^2$ [again a member of $C\{n!\}$ in $(0, \infty)$ and satisfying (3), (4) and (2) with $\nu_n=2n+1$, hence $G=2$] shows that condition (1) can not be removed. It is also shown (theorem 2) that if only (4) holds and $C\{M_n\}$ is quasi-analytic then $\lim |f^{(n)}(0)|^{1/n}=0$. A different type of answer to the problem is obtained by removing the assumption (4), the new requirements being stated in terms of a

modification of the Carleman-Ostrowski condition for quasi-analyticity (Theorem 3). Finally the same problem is answered concerning functions $f(x)$ which are even, of period 2π, infinitely differentiable and belong to $C\{M_n\}$ in $[0, 2\pi]$ (Theorems 4 and 5). The proofs use the connection with Watson's problem via the Laplace transform of $f(x)$ thereby connecting the present problems with earlier results of Mandelbrojt on lacunary power series and Dirichlet series given in his Séries Lacunaires [Actual. Sci. Ind. no. 305, Hermann, Paris, 1936. The exponent of 2 in the last inequality on page 274 should be $3\mu j-j+1$ instead of $3\mu j-j-1$. In formula (13), page 277, the last factor on the right hand side should read $[\pi(1+\epsilon)/G']^{2p+1}$].

I. J. Schoenberg.

Calculus

Taylor, A. E. Derivatives in the calculus. Amer. Math. Monthly **49**, 631–642 (1942). [MF 7813]

Lévy, Paul. À propos du théorème fondamental de la théorie des jacobiens. Enseignement Math. **38**, 218–226 (1942). [MF 7298]

The author criticizes the statement and the proof of the classical theorem to the effect that if the Jacobian of n functions $u_i=f_i(x_1, \cdots, x_n)$ $(i=1, 2, \cdots, n)$ vanishes there must exist a functional relation $\Phi(u_1, u_2, \cdots, u_n)=0$ between the u's. The function Φ should, of course, be continuous and different from zero almost everywhere. Considering the example [due to Laurent Schwartz] of two functions $u_1=x_1 \sin x_1$, $u_2=x_1 \sin \alpha x_1$, where α is irrational, the author points out that the Jacobian is zero everywhere and yet as x_1 varies unrestrictedly the point (u_1, u_2) describes a curve which is everywhere dense (due to the irrationality of α) in the plane. Thus no functional relation of the above type can exist between u_1 and u_2. On the other hand if x_1 is restricted to a finite interval a functional relation can be found. The same situation presents itself in the case $u_1=x_1^{\frac{1}{2}} \sin x_1^{\frac{1}{2}}$ and $u_2=x_1^{\frac{1}{2}} \sin \alpha x_1^{\frac{1}{2}}$ even though all partial derivatives of the first order are bounded in the whole plane. An analysis of these examples (and similar ones) leads to the following restatement of the classical theorem: Let $u_1, u_2, \cdots, u_n$ be continuous functions of $x_1, x_2, \cdots, x_n$ having continuous and bounded partial derivatives of first order and such that their Jacobian vanishes in a closed region R. Then (a) if R is bounded a functional relation between the u's exists; (b) if R is not bounded there need not be a functional relation between the u's. As far as the proof of (a) is concerned the author first notices that the set E described by $(u_1, u_2, \cdots, u_n)$ as $(x_1, \cdots, x_n)$ varies in R is a bounded closed set. Then he shows that the measure of E is 0 and puts $\Phi(u_1, u_2, \cdots, u_n)$ equal to the distance of $(u_1, u_2, \cdots, u_n)$ from E. *M. Kac.*

Vicente Gonçalves, J. Sur les systèmes de fonctions à Jacobien nul. Portugaliae Math. **3**, 157–170 (1942). [MF 7421]

It is shown that if $u=\varphi(x,y)$ and $v=\psi(x,y)$ are differentiable in a domain A, through every point of which at least one of these functions has a level-curve (that is, a continuous curve, possessing a tangent nearly everywhere, on which the function is constant), then a necessary and sufficient condition that the level-curves for either function in A are also level-curves for the other is that the Jacobian vanish identically in A, except possibly at points at which the level-curves fail to have a tangent. In this case u and v are functionally related; the derivative Dv is found to be $\psi_x/\varphi_x=\psi_y/\varphi_y$ at points at which φ_x and φ_y do not vanish simultaneously and otherwise the limit of these fractions at points where these limits exist. Conditions are obtained under which a function $f(x,y)$ possesses level-curves through every point of a domain. The extension of these results to sets of n functions in n variables are indicated.

A. Dresden (Swarthmore, Pa.).

Calugareanu, Georges. Sur la structure des transformations ponctuelles du plan. Mathematica, Timişoara **18**, 68–76 (1942). [MF 7215]

The geometric structure of point transformations in a plane is considered, effected by analytic functions of two real variables. *O. Szász* (Cincinnati, Ohio).

Dwight, Herbert B. Inverse functions of complex quantities. Elec. Engrg. **61**, 850–853 (1942).

The paper concerns itself with the inverse functions of $\sin(x+iy)$, $(x+iy)^{\frac{1}{2}}$, $\log(x+iy)$, etc.

Wylie, C. R., Jr. New forms of certain integrals. Amer. Math. Monthly **49**, 457–461 (1942). [MF 7150]

The author derives, in a direct and simple manner, the values of certain indefinite integrals; the form of the results appears to be new. The integrals discussed are $\int x^n \cos x dx$, $\int x^n \sin x dx$, $\int x^n e^x dx$, $\int x^n e^{-x} dx$, $\int x^n J_m(x) dx$, where $J_m(x)$ is is a Bessel function of order m. The following results illustrate the nature of the formulae found:

$$\int x^n \cos x dx=(-1)^{n/2} n![S_{(n+2)/2}(\cos x) \sin x - S_{n/2}(\sin x) \cos x], \quad n>0, \text{ even},$$

$$\int x^n e^{-x} dx=-n! S_{n+1}(e^x) e^{-x}, \qquad n>0, \text{ integer},$$

$$\int x^{m+2k+1} J_m(x) dx=(-1)2^{m+2k} k!\Gamma(m+k+1) \times\{S_{k+1}(J_m(x)) J_{m+1}(x)-S_k(J_{m+1}(x)) J_m(x)\} x,$$

where $S_r(f(x))$ denotes the rth partial sum of the Maclaurin series for $f(x)$. *M. A. Basoco* (Lincoln, Neb.).

Kopal, Zdeněk. Theoretical light curves of close eclipsing systems. Proc. Amer. Philos. Soc. **85**, 399–431 (1942). [MF 6699]

Kopal, Zdeněk. Theoretical light-curves of close eclipsing systems. II. Astrophys. J. **96**, 20–27 (1942). [MF 7016]

Kopal, Zdeněk. The calculation of rotation factors for eclipsing binary systems. Proc. Nat. Acad. Sci. U.S.A. **28**, 133–140 (1942). [MF 6484]

The author investigates the theoretical light curves of close eclipsing binary systems taking account of the departures of both components from spherical shape due to axial rotation and mutual tidal action. The calculations are carried through the terms of fourth degree in the expansion of the deformation in spherical harmonics. The principal mathematical problem encountered in the investigation is the evaluation of integrals of the types

$$A_n{}^m=\int_s^{r_1}\int_{-(r_1^2-z^2)^{\frac{1}{2}}}^{(r_1^2-z^2)^{\frac{1}{2}}} x^m z^n dx dy,$$

$$B_n{}^m=\int_{\delta-r_2}^{s}\int_{-(r_2^2-(\delta-z)^2)^{\frac{1}{2}}}^{(r_2^2-(\delta-z)^2)^{\frac{1}{2}}} x^m z^n dx dy,$$

where $z^2=r_1^2-x^2-y^2$. These arise in the course of evaluating the surface integral $\int_S J \cos \gamma \, d\sigma$ giving the loss of light during eclipse, J being intensity on the surface of the eclipsed star, γ the angle of foreshortening and S the eclipsed surface. The author finds that A_n^m in all cases and B_n^m for n even are integrable in terms of elementary functions. For n odd B_n^m can be expressed in terms of Legendre's elliptic integrals of the first and second kinds. Using these integrals the author obtains expressions (much too long to be quoted here) giving the light loss for both partial and annular eclipses. The calculation of rotation factors for eclipsing binary systems is also shown to depend upon integrals of the types A_n^m and B_n^m. *W. E. Milne.*

Wilson, R. On the evaluation of

$$\int \frac{dx}{(x-e)^{n+1}\sqrt{(ax^2+2bx+c)}}.$$

Edinburgh Math. Notes no. 32, 13–14 (1941). [MF 7547]

Pollard, W. G. Evaluation of surface integrals by electrical images. Amer. Math. Monthly 49, 604–609 (1942). [MF 7486]

The well-known method of images in electrostatics enables one to determine in certain simple cases the field and its potential energy due to a conductor and a system of given fixed charges. Since the potential energy of the field can also be expressed as a surface integral involving the induced charge distribution which in turn is a function of the field at the surface, the method of images can be used to evaluate such integrals without quadratures. The author evaluates the special integrals obtained from the cases where the conductor is a plane or a sphere. The method appears to have very limited application since one starts with a given electrical system rather than with a given integral.

P. W. Ketchum (Urbana, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Tambs Lyche, R. Une fonction continue sans dérivée. Enseignement Math. 38, 208–211 (1942). [MF 7295]

The construction of the function in question has been described previously [Norske Vid. Selsk. Forh. 12, 45–48 (1939); these Rev. 1, 109].

Lebesgue, Henri. Une fonction continue sans dérivée. Enseignement Math. 38, 212–213 (1942). [MF 7296]

Notes on the paper reviewed above.

Vessiot, Ernest. Sur la variation des fonctions. Enseignement Math. 38, 214–217 (1942). [MF 7297]

In this note the author proves that a function which has a derivative everywhere equal to zero is constant, and that a function with an everywhere positive derivative is increasing. The novelty in the argument consists in obtaining these results without the use of the mean value theorem; instead, a simple argument employing successive subdivision is used. The author says that he is unable to determine whether the argument is actually new.

J. A. Clarkson (Philadelphia, Pa.).

Favard, J. Sur une généralisation de la condition de Lipschitz d'ordre un. Mathematica, Timişoara 18, 26–36 (1942). [MF 7213]

A function whose nth order divided differences have a common bound has, almost everywhere, a bounded nth derivative; if the nth order divided differences are non-negative, the function has almost everywhere a nondecreasing $(n-1)$th derivative. The second result was obtained by T. Popoviciu, who also showed that, if the differences of order n are bounded, then the function has continuous derivatives of every (integral or fractional) order less than n [Mathematica, Cluj 8, 1–85 (1934)]. The author's proofs are very concise. [The results remain true if only differences with equally spaced points are used, provided that the function is assumed bounded [Montel, Bull. Soc. Math. France 46, 151–192 (1918), in particular, p. 183; Popoviciu, op. cit.; Boas and Widder, Duke Math. J. 7, 496–503 (1940); these Rev. 2, 219].] *E. S. Pondiczery.*

Birkhoff, Garrett. Generalized arithmetic. Duke Math. J. 9, 283–302 (1942). [MF 6869]

A number is any nonvoid partially ordered set. The usual concepts of cardinal and ordinal number correspond to the extreme cases of a totally unordered set and a well-ordered set, respectively. The operations of cardinal and ordinal addition, multiplication and exponentiation are defined. In particular, a new definition of ordinal exponentiation is given which is equivalent to that of Hausdorff in the case he considered and under which the family of simply ordered sets is closed. Particular properties of the operations are given for each of several special types of numbers.

N. Dunford (New Haven, Conn.).

Duthie, W. D. Segments of ordered sets. Trans. Amer. Math. Soc. 51, 1–14 (1942). [MF 6087]

The author defines the segment $[x, y]$ joining the elements x and y of a lattice (or more generally of a partially ordered set) to be the set of all elements z such that $xy \leqq z \leqq x+y$, where xy and $x+y$ denote lattice meet and join. The principal object of the paper is to show that many of the important properties of lattices can be conveniently defined and studied in terms of this notion of segment. For example, conditions are derived in terms of segments that a lattice be modular, or distributive, or that two elements be complements. It is shown that the set L_s of all segments of a lattice L is itself a lattice with set inclusion as the ordering relation. The lattice L_s always has a zero element, namely the null-segment, and is complemented if L is. Other relations between L and L_s are given. If L is modular, then L_s is pseudo-modular, and if L is distributive, then L_s is pseudo-distributive. A subset of a lattice is called convex if it contains the segment joining any pair of its elements. Any lattice can be imbedded in the complete lattice consisting of all its convex subsets, a result similar to the MacNeille imbedding theorem. Relations between ideals in a lattice and segments and convex sets are derived. A notion called m-distributivity is defined for lattices with a dimension function, and applied to complemented modular lattices, such as those occurring in the theory of continuous geometries. *O. Frink* (State College, Pa.).

Sanders, S. T., Jr. A linear transformation whose variables and coefficients are sets of points. Bull. Amer. Math. Soc. 48, 440–447 (1942). [MF 6711]

The transformation T_n is $x_i = \sum_{j=1}^{n} a_{ij} x_j'$, $i=1, \ldots, n$, where the a_{ij}, x_j' are sets and the indicated operations are

those of set addition and intersection. The a_{ij} are called independent if, for any system of products P_k of the a_{ij}, $P_h \subset \sum_{k\neq h} P_k$ implies that some factor of P_h is a factor of some P_k for $k\neq h$. The result is: The iterates of T_n consist of at most $(n-1)^2+N$ distinct transformations, where N is the least common multiple of $1, \ldots, n$. If the a_{ij} are independent, the maximum is attained and N of the iterates recur periodically. The method is based on an examination of finite subsequences of the iterates of T_n for monotoneity of the associated matrices, where $\|a_{ij}\| \subset \|a_{ij}'\|$ is defined by $a_{ij} \subset a_{ij}'$ for all i, j. *L. W. Cohen* (Madison, Wis.).

Albuquerque, J. Sur les ensembles clairsemés. Portugaliae Math. 3, 132–156 (1942). [MF 7211]

This paper is essentially a compendium of known results on "separierte Mengen," "ensembles clairsemés" and sets having the property of Denjoy, together with new results such as: In a space $(\mathfrak{S})_\alpha$, which satisfies the condition 3° of F. Riesz and is $\aleph_\alpha$-perfectly separable, all perfectly compact sets possessing the property of Denjoy of order $\aleph_\alpha$ are "clairsemé." A space (F) of character $\aleph_\alpha$ satisfying the condition 2° of F. Riesz of order $\aleph_\alpha$ is said to be a space $(\mathfrak{S})_\alpha$. *J. F. Randolph* (Ithaca, N. Y.).

de Mira Fernandes, A. La condizione (N) di Lusin et le condizioni (T) ed (S) di Banach. Condizioni (S_1) ed (S_2). Portugaliae Math. 3, 120–123 (1942). [MF 7087]

Let I be an interval, (A) the class of measurable sets $A\subset I$, $F(x)$ a finite real function on I, $F[A]$ the image of A, and $|A|$, $|F[A]|$ the measures of A, $F[A]$, respectively. The author calls $|F[A]|$ uniformly continuous over (A) if $F[0]=0$, by convention, and $||F[A_2]|-|F[A_1]||<\epsilon$ if $|A_2-A_1|<\delta_\epsilon$. Applying known theorems of Bary and Menchoff, he remarks that the uniform continuity of $|F[A]|$ over (A) follows if either $F(x)$ is the superposition of two absolutely continuous functions or $F(x)$ is continuous and differentiable except on a set of measure zero.
L. W. Cohen (Madison, Wis.).

Stone, A. H. and Tukey, J. W. Generalized "sandwich" theorems. Duke Math. J. 9, 356–359 (1942). [MF 6873]

Ulam has shown that any three sets in space, each of finite outer measure, may be bisected (in the sense of measure) by a plane. The authors generalize this to the case of n subsets of any set R with a Carathéodory outer measure. The notion of a plane is replaced by a suitably restricted real function $f(\Lambda, x)$ on $S^n\times R$ (S^n is the n-sphere). *N. Dunford* (New Haven, Conn.).

Linés Escardó, E. Measure of a set transformed from another of given measure. Revista Mat. Hisp.-Amer. (4) 2, 67–71 (1942). (Spanish) [MF 7130]

Let C be a measurable point set in Euclidean n-space, and C' the image of C under a transformation

$$x_i'=f_i(x_1, x_2, \cdots, x_n).$$

Under the supposition that the functions f_i have continuous partial derivatives and that the Jacobian

$$J=\partial(f_1f_2\cdots f_n)/\partial(x_1x_2\cdots x_n)$$

does not vanish identically, the author shows that C' is measurable, and that the measure of C' is equal to the Lebesgue integral of J over the set C. He also considers the question of change of variable in an absolutely continuous additive set function. *H. S. Wall* (Evanston, Ill.).

Chelidze, W. Über Funktionen eines Intervalls. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 9, 1–17 (1941). (Russian. German summary) [MF 7375]

The principal aim of this paper is to prove the following theorem: If the function $F(x, y)$ is absolutely continuous (in a special sense) on the rectangle r_0 and if the approximate derivative $D_{ap}F(x, y)$ exists and vanishes almost everywhere, then $F(x, y)=\varphi(x)+\psi(y)$. Here

$$D_{ap}F(x_0, y_0)=\lim \text{ap}\frac{F(x, y)+F(x_0, y_0)-F(x_0, y)-F(x, y_0)}{(x-x_0)(y-y_0)}.$$

A false version of this theorem in a previous paper by the author [same Trav. 2, 109–142 (1937)] is corrected here by altering the definition of absolute continuity for $F(x, y)$. Several definitions are considered and their relations with previous definitions by Krzyżański and Kempisti determined. For the theorem above one of the new definitions is chosen. It is somewhat long to state. *J. V. Wehausen.*

Jeffery, R. L. Perron integrals. Bull. Amer. Math. Soc. 48, 714–717 (1942). [MF 7270]

The purpose of this note is to give a proof of the integration by parts theorem for the Perron integral, using directly the definition of the Perron integral. This theorem has been proved by Saks [Théorie de l'Intégrale, Monografie Matematyczne, vol. 2, Warsaw, 1933, p. 201] in the form

$$\int fg=Fg\Big]_a^b-\int_a^b Fdg,$$

with g of bounded variation and Denjoy integrable and so Perron integrable and with $F(x)=\int_a^x f$, using the properties of the Denjoy integral. This note treats, by the method of Perron integration, first the case where $g(x)$ is continuous on (a, b) and $g'(x)$ is finite except for a denumerable set and summable, in which case the term $\int_a^b Fdg$ becomes $\int_a^b Fg'$, and then the case where $g(x)$ is of bounded variation but $g'(x)$ is limited to be finite except for a denumerable set.
T. H. Hildebrandt (Ann Arbor, Mich.).

McShane, E. J. On Perron integration. Bull. Amer. Math. Soc. 48, 718–726 (1942). [MF 7271]

The objective of this note is the same as that of Jeffery above. The present note derives another definition of the Perron integral [presented to Oslo Congress in 1936] in which the major and minor functions are each replaced by major right and left, and minor right and left functions, respectively, $f(x)$ being P^* integrable on (a, b) when for every ϵ there exists a set of four such functions $\varphi^i(x)$ such that $|\varphi^i(b)-\varphi^j(b)|<\epsilon$, for $i, j=1, \cdots, 4$. The properties of the Perron integral are easily transferred to the P^* integral and it is possible to prove the integration by parts theorem in the general form stated by Saks. The note closes by showing that the Perron and P^* integrability are the same and the integrals agree, so that the note proves the general Saks result by Perron methods. *T. H. Hildebrandt.*

Radó, T. and Reichelderfer, P. Convergence in length and convergence in area. Duke Math. J. 9, 527–565 (1942). [MF 7335]

The writers generalize certain results of Adams and Clarkson [Bull. Amer. Math. Soc. 40, 413–417 (1934)] and Adams and Lewy [Duke Math. J. 1, 19–26 (1935)] concerning convergence in length and convergence in variation first to vector functions of one variable, then to ordinary functions of two variables and finally to vector functions

of two variables. We shall describe only the last two sets of results, those for curves being easily formulated by the reader.

Let $\{f_n(x, y)\}$, $n=0, 1, 2, \cdots$, all be defined, continuous and of bounded variation in the sense of Tonelli (B.V.T.) on the cell $R:[a, c; b, d]: a\leqq x\leqq b, c\leqq y\leqq d$, and suppose that $f_n(x, y)$ converges uniformly to $f_0(x, y)$ on R. Let the x and y variations $T_x(f)$ and $T_y(f)$ be defined by

$$T_x(f)=\int_c^d V_x(y, f)dy, \quad T_y(f)=\int_a^b V_y(x, f)dx,$$

where $V_x(y, f)$ denotes the variation of $f(x, y)$ as a function of x alone (lower semi-continuous in y), for instance; and let $A(f)$ denote the Lebesgue area of the surface $z=f(x, y)$. Then they say that (1) $f_n-v\to f_0$ if $T_x(f_n)\to T_x(f_0)$, $T_y(f_n)\to T_y(f_0)$ (convergence in variation); (2) $f_n-sv\to f_0$ if $T_x(f_n-f_0)\to 0$, $T_y(f_n-f_0)\to 0$ (strong convergence in variation); (3) $f_n-a\to f_0$ if $A(f_n)\to A(f_0)$ (convergence in area). The bulk of their results for this case are as follows: (i) if $f_n-a\to f_0$, then $f_n-v\to f_0$; (ii) if $f_n-a\to f_0$, and f_0 is A.C.T. (absolutely continuous in the sense of Tonelli), then $f_n-sv\to f_0$; (iii) if $f_n-sc\to f_0$, each f_n being A.C.T., then f_0 is A.C.T.

In case of vector functions $X(u, v)$, we replace the class B.V.T. above by the class A of vectors X which merely possess partial derivatives almost everywhere and for which $A(X)$ is finite, $A(X)$ being the Lebesgue area of the surface $x=X(u, v)$. The class A.C.T. is replaced by the subclass A.J. consisting of those of the above for which $A(X)$ is given by the usual double integral. Then, if X is of class A, each of the X_n is of class A.J. and

$$A(X_n)\to\iint_R (EG-F^2)^{\frac{1}{2}}dudv<\infty,$$

then

$$\iint_R \left|\frac{\partial(x_n, y_n)}{\partial(x, y)}\right|dudv\to\iint_R \left|\frac{\partial(x, y)}{\partial(u, v)}\right|dudv,$$

etc., and X is also of class A.J. *C. B. Morrey, Jr.*

Theory of Functions of Complex Variables

Zygmund, A. On the convergence and summability of power series on the circle of convergence. II. Proc. London Math. Soc. (2) 47, 326–350 (1942). [MF 7407]

This paper is a continuation of another paper which appeared under the same title in the Fund. Math. 30, 170–196 (1938). Both of them treat the problem of the behaviour of power series of the class H^λ on the circle of convergence. While the first part was concerned with the case $\lambda\geqq 1$ the present paper deals with the case $0<\lambda<1$. It has been shown by Hardy and Littlewood and by Gwilliam [J. London. Math. Soc. 10, 248–253 (1935)] that if $F(z)=\sum_0^\infty c_nz^n$ is analytic for $|z|<1$ and if for some $0<\lambda<1$ the integral $\int_{-\pi}^{+\pi}|F(re^{i\theta})|^\lambda d\theta$ remains bounded when $r\to 1$ (that is, if F belongs to the class H^λ), the series $\sum_0^\infty c_ne^{in\theta}$ is summable $(C, \alpha+\epsilon)$ for almost all θ, where $\alpha=(1/\lambda)-1$ and $\epsilon>0$ is arbitrarily small. It is also known that the result fails for $\epsilon<0$. The main theorem of the present paper is that the result holds for $\epsilon=0$. The method used to prove this theorem, which lies deep, resembles that used to prove the following theorem: If $f(x)$ is a periodic function belonging to the class L and $s_n(x)$ denotes the nth partial sum of its Fourier series, then, q being any positive number, $\sum_{\nu=0}^n|s_\nu(x)-f(x)|^q=o(n)$ for almost all x. This result has been known for a long time for functions $f(x)$ belonging to the class L^r $(r>1)$ but it is only recently that Marcinkiewicz [J. London Math. Soc. 14, 162–168 (1939)] succeeded in proving that the result was true for $f\epsilon L$ and for $q=2$ (hence for $q\leqq 2$). Here the result is shown to be true for every positive q by an argument entirely different from the argument used by Marcinkiewicz. *R. Salem* (Cambridge, Mass.).

Manning, Rhoda. On the derivatives of the sections of bounded power series. Ann. of Math. (2) 43, 617–622 (1942). [MF 7394]

Let F be the class of functions $f(z)=a_1z+a_2z^2+\cdots$ which are regular and bounded by 1 in the unit circle. Let $|z|\leqq R_n$ be the largest circle with center at the origin in which $|S_{n+1}'(z)|\leqq 1$ for all functions $f(z)\epsilon F$. It is shown that if n is an even integer, $n\geqq 12$, then R_n is the smallest positive root of the equation

$$1-2r-r^2+(-1)^n\{(2n+4)r^{n+1}+(2n+2)r^{n+2}\}=0.$$

This had been proved by Schur and Szegö in case n is an odd integer, $n\geqq 1$. It follows that the limit of R_n as n becomes infinite is $\sqrt{2}-1$. Also, for every even n, $n\geqq 12$, the partial sums of $f'(z)$ are in general bounded by 1 in a circle larger than that for $f'(z)$ itself. The behavior of R_n for even integers less than 12 is different from the behavior for even integers equal to or greater than 12.

A. C. Schaeffer (Stanford University, Calif.).

Siegel, Carl Ludwig. Iteration of analytic functions. Ann. of Math. (2) 43, 607–612 (1942). [MF 7392]

Let $f(z)=\sum_{k=1}^\infty a_kz^k$ be a convergent power series (that is a series with positive radius of convergence). There is exactly one formal (convergent or divergent) solution $\varphi(\rho)=\rho+\cdots$ of Schröder's functional equation $\varphi(a_1\rho)=f(\varphi(\rho))$, and φ is called the Schröder series of f. The author solves the problem of the existence of a number a_1 (of absolute value 1) such that every convergent power series $f(z)=a_1z+\cdots$ has a convergent Schröder series by proving that if $\log|a_1^n-1|=O(\log n)$ as $n\to\infty$, a condition which holds for almost all points of the unit circumference $|a_1|=1$, then the Schröder series is always convergent.

D. C. Spencer (Stanford University, Calif.).

Pompeiu, D. Les définitions de l'holomorphie et le prolongement analytique. Mathematica, Timişoara 18, 112–124 (1942). [MF 7216]

Some remarks of didactical nature. *S. Mandelbrojt.*

Biggeri, Carlos. A general theorem referring to the Julia lines of entire functions. Bol. Mat. 15, 89 (1942). (Spanish) [MF 7221]

The author announces a rather complicated sufficient condition, involving the elliptic modular function, for a line to be a Julia line. *E. S. Pondiczery.*

Spencer, D. C. Some remarks concerning the coefficients of schlicht functions. J. Math. Phys. Mass. Inst. Tech. 21, 63–68 (1942). [MF 7247]

Let the function

(1) $$f_s(z)=z+c_{s+1}z^{s+1}+\cdots+c_{ns+1}z^{ns+1}+\cdots$$

be regular and univalent for $|z|<1$. Let

$$(2)\quad f_s^{(0)}(z)=z(1-z)^{-2/s}=z+a_{s+1}^{(0)}z^{s+1}+\cdots+a_{ns+1}^{(0)}z^{ns+1}+\cdots.$$

The conjectured inequality

$$(3)\qquad |c_{ns+1}|\leqq a_{ns+1}^{(0)},\qquad n=1,2,\cdots,$$

is known to be true for $n=2$, $s=1$, and also for general s when $n=1$. Fekete and Szegö have established the existence of functions (1) for which (3) is false for $n=2$, $s>1$.

The author constructs explicitly, for any $n>1$, functions f_2 for which $|c_{2n+1}|>a_{2n+1}^{(0)}=1$. Analogous constructions fail to disprove the Bieberbach conjecture $|c_n|\leqq a_n^{(0)}=n$ for any $n>1$. He shows that infinitesimal deformations of $f_2^{(0)}$ increase $|a_{2n+1}^{(0)}|$ but infinitesimal deformations of $f_1^{(0)}$ decrease $|a_n^{(0)}|$. Therefore a local maximum of $|c_n|$ occurs at $f_1=f_1^{(0)}$. K. Joh [Proc. Phys.-Math. Soc. Japan (3) **20**, 591–610 (1938)] had previously shown that $|c_4|$ has a local maximum at $f_1^{(0)}$ [Proc. Phys.-Math. Soc. Japan (3) **23**, 409–423 (1941); these Rev. **3**, 201]. Now $\zeta=\zeta(h,\eta,z)$, $h\geqq 0$, $|\eta|=1$, η complex, is determined from the equation

$$\zeta\cdot(1+\eta\zeta)^{-2}=\alpha z\cdot(1+\eta z)^{-2},\qquad \alpha=(1+h)^{-1}.$$

Let

$$\zeta_s^{(j)}=\{\zeta(h_j,\eta_j,z^s)\}^{1/s}=\sum_{n=0}^{\infty}b_{ns+1}^{(j)}z^{ns+1},\quad j=1,2,\cdots,k.$$

Define

$$f_s^{(1)}(z)=(1/b_1^{(1)})[f_s^{(0)}\{\zeta_s^{(1)}(z)\}],\quad\cdots,$$
$$f_s^{(k)}(z)=(1/b_1^{(k)})[f_s^{(k-1)}\{\zeta_s^{(k)}(z)\}]$$
$$=z+a_{s+1}^{(k)}z^{s+1}+\cdots+a_{ns+1}^{(k)}z^{ns+1}+\cdots.$$

J. Basilewitsch [Rec. Math. [Mat. Sbornik] N.S. **1** (**43**), 211–228 (1936)] has shown that, given $\epsilon>0$, $|c_n-a_n^{(k)}|<\epsilon$ for $k<k_0(\epsilon,f_1)$ and suitable parameters h_j, η_j. The author obtains the explicit formula

$$[\partial a_{ns+1}^{(k)}/\partial h_j]_{h_1=h_2=\cdots=h_k=0}$$
$$=-na_{ns+1}^{(0)}-(2/s)\sum_{\mu=0}^{n-1}(\mu s+1)a_{\mu s+1}^{(0)}\eta_j^{n-\mu},$$

from which he is able to obtain ($\eta_j=e^{i\theta}$)

$$\Re[\partial a_n^{(k)}/\partial h_j]_{h_1=h_2=\cdots=h_k=0}<0,\quad s=1,\ n>1,\ \theta\neq\pi,$$
$$\Re[\partial a_{2n+1}^{(k)}/\partial h_j]_{h_1=h_2=\cdots=h_k=0}>0,\qquad s=2,\ n>1,$$

for $2\pi/(n+1)<|\theta|<2\pi/(n+1)(1+1/n)$.

M. S. Robertson (New Brunswick, N. J.).

Robinson, Raphael M. Bounded univalent functions. Trans. Amer. Math. Soc. **52**, 426–449 (1942). [MF 7455]

Suppose that $f(z)$ is regular and univalent for $|z|<1$ with $|f(z)|<1$, $f(0)=0$, and for any fixed $z\neq 0$, $|z|<1$, let $a=|f'(0)|$, $b=|z|$, $c=|f(z)|$, $d=|f'(z)|$. G. M. Golusin [Rec. Math. [Mat. Sbornik] N.S. **1** (**43**), 127–135 (1936)] showed that known inequalities relating a, b, c, d may be derived in a striking way from Löwner's differential equation. The author here systematizes and extends this method, and for each subset of the four quantities a, b, c, d obtains a sharp inequality relating the members of this subset. In addition to known results, interesting new inequalities are obtained, and in particular, as limiting cases, new results for unbounded univalent functions. For example, let $F(z)$ be regular and univalent for $|z|<1$, $F(0)=0$, $F'(0)=1$. Then if $|F(z)|\leqq\frac{1}{4}$, $|F'(z)|\leqq 1+3\cdot 2^{-3/2}=2.06\cdots$. But if $|F(z)|>\frac{1}{4}$, no upper bound for $|F'(z)|$ exists.

D. C. Spencer (Stanford University, Calif.).

Weyl, Hermann and Weyl, Joachim. On the theory of analytic curves. Proc. Nat. Acad. Sci. U.S.A. **28**, 417–421 (1942). [MF 7288]

The authors make a series of observations which indicate how the methods developed by Ahlfors in the theory of meromorphic curves [Acta Soc. Sci. Fennicae. Nova Ser. A **3**, no. 4 (1941); these Rev. **2**, 357] can be adapted to the theory of arbitrary analytic curves whose parameter varies over a given Riemann surface, the latter theory having been inaugurated by J. Weyl [Ann. of Math. (2) **42**, 371–408 (1941); these Rev. **3**, 81]. It is shown how this adaptation permits the general theory of analytic curves to be completed to the same degree as the more special theory of meromorphic curves. The observations, couched in the language of electrostatics, indicate the technical difficulties of the problem and how they can be overcome. It is stated that a more detailed account will probably be published in a monograph in the Annals of Mathematics Studies.

M. H. Heins (Chicago, Ill.).

Kwesselawa, D. Zur Theorie der konformen Abbildungen. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **9**, 19–32 (1941). (Russian. German summary) [MF 7376]

The author studies the variation of a function representing a domain on the unit circle if the domain changes slightly. Two definitions for the distance of domains are mentioned: (i) Two domains D, D_1, whose boundaries are closed Jordan curves Γ, Γ_1, respectively, have the distance ϵ if it is the smallest number such that every point of Γ has a distance from Γ_1 less than ϵ and, similarly, every point of Γ_1 has from Γ a distance less than ϵ. (ii) If $\rho(\varphi)$ and $\rho_1(\varphi)$ are the equations in polar coordinates of Γ and Γ_1 respectively, then D and D_1 have the distance

$$\epsilon=\max_{0\leqq\varphi\leqq 2\pi}|\rho(\varphi)-\rho_1(\varphi)|.$$

If $\rho(\varphi)$, $\rho_1(\varphi)$ are multiple valued, it is obvious how to adapt the definition to this more general case. We define $w=f(z)$ and $w=f_1(z)$ as the functions which represent D and D_1 on the unit circle and we norm them by the conditions: $f(0)=f_1(0)=0$, $f'(0)>0$, $f_1'(0)>0$. Then in an older paper A. Markuševič [Rec. Math. [Mat. Sbornik] N.S. **1** (**43**), 885–886 (1936)] showed that, under the first definition of the distance of D and D_1, and certain conditions on Γ and Γ_1, we have $|f(z)-f(z_1)|<k\epsilon\log 1/\epsilon$. The present paper's main theorem is as follows: if D and D_1 are two star domains with respect to $w=0$ which have the distance ϵ according to (ii), then the above normed functions $f(z)$ and $f_1(z)$ satisfy $|f(z)-f_1(z)|<M(r)\epsilon$ for $|z|\leqq r<1$. Here $M(r)$ depends only on r. The proof given would require a slight modification in order to show that $M(r)$ is independent of D and D_1. This theorem is supplemented by another one showing that the above theorem is false if we adopt the more general definition (i) for the distance of two domains.

František Wolf (Berkeley, Calif.).

Heins, Maurice H. On the continuation of a Riemann surface. Ann. of Math. (2) **43**, 280–297 (1942). [MF 6463]

Let F be a Riemann surface. If there exists another Riemann surface G such that F admits a 1 : 1 conformal

map onto a proper part F' of G, then F is said to be continuable and G is a continuation of F; otherwise F is said to be maximal, as in the case of a closed Riemann surface. Open maximal surfaces have been shown by Radó to exist, and a proof has been given by Bochner, using transfinite induction, that every continuable Riemann surface has a maximal continuation.

The author studies the problem of the continuation of F through an analysis of the Fuchsian groups associated with the uniformizing functions for F and its extensions. If $w=w(w_0, z)$ is the function mapping the covering surface of F on $|z|<1$ with $w=w_0$ going into $z=0$, w is automorphic with respect to a certain Fuchsian group $\mathfrak{G}(F)$ of linear transformations. A space F^* is formed by identifying the points of $|z|<1$ which are equivalent with respect to $\mathfrak{G}(F)$; F^* can be seen to be a Riemann surface conformally equivalent to F; F^* is written $(|z|<1)$ (mod $\mathfrak{G}(F)$). If F is continuable and G is a continuation of F, two Fuchsian groups $\mathfrak{G}(F)$ and $\mathfrak{G}(G)$ are determined, and the function determining the continuation sets up a homomorphism of $\mathfrak{G}(F)$ in $\mathfrak{G}(G)$. Now the homomorphic image of $\mathfrak{G}(F)$ may not exhaust $\mathfrak{G}(G)$. Let Φ denote the class of F and all its continuations, and Φ_0 the class of F and its continuations which give a strict homomorphic map of $\mathfrak{G}(F)$ on $\mathfrak{G}(G)$. Now Φ_0 can be regarded as an L-space of Fréchet and can be shown to be compact. From the compactness follows the theorem that every continuable F admits a maximal continuation, and without the use of transfinite induction.

The author now proceeds to investigate the structure of the homomorphism of $\mathfrak{G}(F)$ on $\mathfrak{G}(G)$ and to deduce necessary and sufficient conditions that F be continuable. The group $\mathfrak{G}(F)$ is said to be of the first or second kind according as the set of limit points of $\{T(z_0)\}$, where $|z_0|<1$ and $T\epsilon\mathfrak{G}(F)$, consists of $|z|=1$ or a totally disconnected set on $|z|=1$. The following theorem is obtained: If $\mathfrak{G}(F)$ is of the second kind, F is always continuable. If $\mathfrak{G}(F)$ is of the first kind, F is continuable if and only if F admits a boundary element of the first type [de Kerékjártó, Topologie, Springer, Berlin, 1923]. In case F has no boundary element of the first type, the author is able to exhibit a continuation of F; in fact, such a continuation is given by

$$G=(\text{extended } z\text{-plane}-E)(\text{mod } \mathfrak{G}(F)),$$

where E is the set of points on $|z|=1$ where $\mathfrak{G}(F)$ is not properly discontinuous.

The last section of the paper is devoted to the derivation of a special type of maximal continuation G, namely, such that in G there is a $(1:1)$ directly conformal image of F which is dense in G. This kind of maximal continuation is shown to exist for every continuable F. *J. W. Green.*

Blanc, Ch. Complexes à n dimensions et intégrales abéliennes. Comment. Math. Helv. **14**, 212–229 (1942).

In previous papers [Comment. Math. Helv. **12**, 153–163 (1939); **13**, 54–67 (1940); see these Rev. **1**, 213; **2**, 293], for functions defined on plane networks of points the author has defined and investigated analogues of harmonic functions and of analytic functions of a complex variable. In particular, for infinite nets he has considered and obtained partial results for a problem of that type. It is natural to consider also functions defined at the vertices of finite polyhedra, analogous to algebraic Riemann surfaces. For these the author gives a development analogous to the theory of Abelian integrals, obtaining especially an analogue of the Riemann-Roch theorem. *E. F. Beckenbach.*

Schiffer, Menahem. Sur la variation du diamètre transfini. Bull. Soc. Math. France **68**, 158–176 (1940). [MF 6813]

The problem of Pólya-Grötzsch consists of finding a closed set C containing n fixed points such that the transfinite diameter $d(C)$ of C is a minimum. The author deals with this problem by a variational method illustrated in the first nontrivial case $n=3$. If the given points are a_1, a_2, a_3 the variation

$$z^*=z+\epsilon e^{i\varphi}(z-a_1)(z-a_2)(z-a_3)(z-z_0)^{-1}$$

is considered; here ϵ, φ are arbitrary real, $\epsilon\rightarrow 0$, z_0 is exterior to the set C for which the minimum is attained. By using a lemma of Fekete-Leja the differential equation

$$\xi^2[f'(\xi)]^2[f(\xi)-(a_1+a_2+a_3+2\alpha)] \\ =[f(\xi)-a_1][f(\xi)-a_2][f(\xi)-a_3]$$

is derived; here $z=f(\xi)=\xi-\alpha+b\xi^{-1}+\cdots$ is the normalized map-function of the exterior of C onto the circle $|\xi|>d(C)$. This set C consists of three analytic arcs meeting at the point $a_1+a_2+a_3+2\alpha$ and forming the angles 120°, a result which was known to Grötzsch. The same variational method is used for a new approach to the inequality $d(A+B)\leqq d(A)+d(B)$ (A and B are two sets with common points situated in the closed interior and exterior of a Jordan curve, respectively). This problem was treated by the author in an earlier paper [Proc. Cambridge Philos. Soc. **37**, 373–383 (1941); these Rev. **3**, 73]. *G. Szegö.*

***Sewell, W. E. Degree of Approximation by Polynomials in the Complex Domain.** Annals of Mathematics Studies, no. 9. Princeton University Press, Princeton, N. J., 1942. ix+236 pp. $3.00.

This is primarily a systematically integrated account of recent research by the author, or by the author in collaboration with J. L. Walsh, on the subject indicated by the title of the book. As a part of a larger theory it is a sequel to Walsh's Interpolation and Approximation by Rational Functions in the Complex Domain [American Mathematical Society Colloquium Publications, vol. 20, 1935]; but in itself it is an independent development of a separate phase of the theory. Let C be an analytic Jordan curve in the z-plane, and let E be the closed point set consisting of C and the region interior to C. Let $f(z)$ be a function which is analytic inside C and continuous on E, and which furthermore satisfies a Lipschitz condition of order α as a function of arc length on C, $0<\alpha\leqq 1$. Then there exist polynomials $p_n(z)$, each of degree indicated by its subscript, such that

$$|f(z)-p_n(z)|\leqq M/n^\alpha$$

on E, the constant M being independent of n and z. If $f(z)$ has a kth derivative satisfying the hypothesis stated, n^α in the last inequality can be replaced by $n^{k+\alpha}$. This is an example of a "direct theorem" relating to "Problem α", according to the terminology which is used in classifying the results. Hypothesis and conclusion can be interchanged, both for $k=0$ and for positive k, provided that $0<\alpha<1$; the statement requires modification for $\alpha=1$. This is an "indirect theorem" under Problem α. "Direct" theorems, that is to say, infer the existence of specified degrees of approximation from properties of continuity and differentiability on the boundary, and "indirect" theorems infer the latter properties from degree of approximation. The theorems stated are special cases of more general formulations in the text. It is of course to be borne in mind, but need not be repeated on every occasion, that approximation

on the curve implies at least the same degree of approximation in its interior.

Let the exterior of the curve C in the z-plane be mapped conformally on the exterior of the circle $|w|=1$ in the w-plane so that the points at infinity correspond to each other, and let C_ρ be the image in the z-plane of the circle $|w|=\rho$, $\rho>1$. If $f^{(k)}(z)$ satisfies a Lipschitz condition of order $\alpha<1$ on the curve C_ρ, for a specified value of $\rho>1$, $f(z)$ can be approximated on C by polynomials of the nth degree with an error not exceeding a quantity of the order of $\rho^{-n}n^{-k-\alpha}$; and a theorem of converse type holds with replacement of $\rho^{-n}n^{-k-\alpha}$ by $\rho^{-n}n^{-k-\alpha-1}$, the description of the behavior of $f(z)$ on C_ρ remaining unchanged. These are direct and indirect theorems under "Problem β." The latter problem is characterized by simultaneous consideration of the loci C and C_ρ. An analytic Jordan curve has been mentioned as boundary for simplicity of statement. An important part of the theory at the stage of development represented by the present account is the extension or adaptation of the results to boundaries of more general character, formed by non-analytic Jordan curves or by several curves exterior to each other. The underlying systematic principle of procedure, varied in its application according to circumstances, is to establish results for the unit circle, and carry them over to more general types of boundary with the aid of conformal mapping. Essential use is made of relations between behavior of the function along the boundary and its behavior on approach to the boundary from the interior of the region. In terms of the latter type of behavior, the significance and validity of theorems under Problem β are extended to negative values of k. Theorems on uniform approximation are accompanied by a corresponding discussion of approximation in the mean, according to the criterion of a contour integral (or, in some of the exercises, an area integral) involving a norm function and a power of the absolute value of the error. Auxiliary topics of which essential use is made in the development of the main theory and which are at the same time of major independent interest and importance are a generalization for the complex domain of the theorems of Markoff and S. Bernstein on the relation between the modulus of a polynomial and that of its derivative, in the form of a comprehensive theorem which includes results of both types together with a continuous transition from one to the other; and the theory of "equally distributed points" on a curve and interpolation based on these points.

There is evidence of composition under pressure of time in minor errors or inaccuracies of expression, by which the reader should not be unduly disturbed. It is the reviewer's impression that these are rather numerous, but not serious in general effect. The detailed exposition is supplemented by exercises and a sketch of open problems in connection with each chapter, and by a bibliography. *D. Jackson.*

Theory of Series

Bleick, W. E. Symmetric relations between the coefficients of reversed power series. Philos. Mag. (7) 33, 637–638 (1942). [MF 7226]

Formulas are given for the first twelve coefficients of the power series $x=f(y)$, where y is a power series in x with given coefficients. *P. W. Ketchum* (Urbana, Ill.).

Edmonds, Sheila M. On the multiplication of series which are infinite in both directions. J. London Math. Soc. 17, 65–70 (1942). [MF 7313]

A series (*) $\sum_{-\infty}^{+\infty}a_n$, infinite in both directions, is said to be convergent if both series $\sum_{-\infty}^{-1}a_n$ and $\sum_0^\infty a_n$ are convergent. Denoting by A' and A'' the sums of the latter two series, we define the sum A of (*) as $A'+A''$. The product of the series (*) $\sum_{-\infty}^{+\infty}a_n$ and $\sum_{-\infty}^{+\infty}b_n$ is defined as the series (***) $\sum_{-\infty}^{+\infty}c_n$, where $c_n=\sum_{\nu=-\infty}^{+\infty}a_\nu b_{n-\nu}$ (the definition presupposes that the series defining c_n are convergent for all n). The main result of the paper is that the three series (*), (**), (***) may be convergent and yet their sums A, B, C need not satisfy the relation $C=AB$.

A. Zygmund (South Hadley, Mass.).

Walsh, C. E. Note on an analogue of Mercer's theorem. J. London Math. Soc. 17, 13–17 (1942). [MF 6969]

The author establishes the following results which, as he shows, include Bosanquet's analogue for absolute convergence of Mercer's theorem [Bosanquet, J. London Math. Soc. 13, 177–180 (1938)] as well as more general theorems. Let $\{a_n\}$ and $\{y_n\}$ be sequences of complex numbers, and let $t_n-a_nt_{n-1}=(1-a_n)y_n$ $(n=N, N+1, \cdots)$; let $|1-a_n|\leqq K(1-|a_n|)$, where K is a fixed positive number. Then $\sum|\Delta y_n|<\infty$ implies $\sum|\Delta t_n|<\infty$; if $a_n\neq 0$ and $\sum|\Delta y_n|<\infty$, then there is a constant l such that

$$\sum|\Delta\{t_n-l(\prod_{\nu=N}^{n}a_\nu)^{-1}\}|<\infty.$$

R. P. Boas, Jr. (Chapel Hill, N. C.).

Hildebrandt, T. H. Remarks on the Abel-Dini theorem. Amer. Math. Monthly 49, 441–445 (1942). [MF 7146]

The author states theorems about series called by Knopp the theorems of Abel and Dini and the theorem of Dini; a concise proof is given. There follows a number of penetrating remarks on conclusions that can be drawn from the quoted theorems. There are several specific generalizations of the type where s_n replaces n in well-known results.

T. Fort (Bethlehem, Pa.).

Fuchs, W. H. J. A note on convergence factors. Proc. Edinburgh Math. Soc. (2) 7, 27–30 (1942). [MF 7491]

In this note $\sum_{n=1}^\infty u_n$ denotes a divergent series of positive decreasing terms for which $\lim_{n\to\infty}u_n=0$. Let $\epsilon_1, \epsilon_2, \cdots$ be real numbers such that $\sum_{\nu=1}^\infty \epsilon_\nu u_\nu$ is convergent. Let $t_n=\sum_{\nu=1}^n \epsilon_\nu$ and $\sigma_n=n^{-1}\tau_n$. Two theorems are proved relative to σ_n. *T. Fort* (Bethlehem, Pa.).

Fort, Tomlinson. Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas. Bull. Amer. Math. Soc. 48, 567–574 (1942). [MF 7050]

Fort, Tomlinson. An addition to "Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas." Bull. Amer. Math. Soc. 48, 949 (1942). [MF 7520]

Let α_n denote, generically, a polynomial of degree n if n is an integer not less than 0, and 0 if n is a negative integer. Let P and Q be linear operators which carry α_n into α_{n-1} and α_{n-k}, respectively, k being a fixed integer not less than 0. Let P and Q have "indefinite inverses" P^{-1} and Q^{-1}, the transforms $P^{-1}f$ and $Q^{-1}f$ being unique except for additive α_0 and α_{k-1}, respectively. Let $f_n(x)$ be a sequence of polynomials and ${}_nL_j$ a double sequence of constants such that

$f_0(x)=1$, $f_n(x)$ has degree n, $Pf_n=nf_{n-1}$, and, when $n\geqq k$,

$$(1)\qquad Q^{-1}k!\binom{n}{k}f_{n-k}=\sum_{j=0}^{k}\binom{n}{j}{}_nL_j f_{n-j}+\alpha_{k-1}.$$

Let $PQ^{-1}f_{n-k}\equiv Q^{-1}Pf_{n-k}+\alpha_{k-1}$. Then ${}_nL_j$ is shown to be independent of n, and the constants are denoted by L_0, L_1, $\cdots$. Let g_0, g_1, $\cdots$ be the sequence of constants obtained by solving the system of equations "$f_n(g)=L_n$" which result from writing the equations $f_n(g)=L_n$ and converting the exponents on g into subscripts. Let

$$(2)\qquad F_n(x)=\sum_{j=0}^{n}\binom{n}{j}f_j(g)f_{n-j}(x),$$

where $f_j(g)$ means L_j, and let the right side of (2) be represented by the symbol $f_n(x+g)$. In terms of these symbols, generalizations of the Euler-Maclaurin summation formula are given. Specializations of P, Q and f_n lead to the ordinary and generalized numbers and polynomials of Bernoulli, Euler and Appell. The g's are the numbers and the F's are the polynomials. *R. P. Agnew* (Ithaca, N. Y.).

Cesco, R. P. On the theory of linear transformations and the absolute summability of divergent series. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Series 2: Revista 2, 147–156 (1941). (Spanish) [MF 7134]

Let A be a matrix with real or complex elements $a_{n,k}$ (n, $k=0, 1, 2, \cdots$). A series $\sum u_n$ with partial sums s_n is absolutely summable A (summable $|A|$) if $\sum|\alpha_n-\alpha_{n-1}|<\infty$, where $\alpha_n=\sum_{k=0}^{\infty}a_{nk}s_k$. Let $\{S\}$ and $\{A\}$ denote, respectively, the classes of series which are absolutely convergent and summable $|A|$. The matrix A is absolutely regular if $\{A\}\supset\{S\}$. Let $\{A^*\}$ denote the class of series $\sum u_n$ summable $|A|$ which remain summable $|A|$ whenever a finite number of zeros is prefixed, and let $\mathfrak{C}$ denote the class of matrices A for which $\{A^*\}=\{A\}$. Each absolutely regular Nörlund matrix (N, p_n) belongs to class $\mathfrak{C}$. For the Riesz matrices (R, q_n), sufficient conditions for $(R, q_n)\epsilon\mathfrak{C}$ are given. A necessary and sufficient condition that the Cauchy product of the series $\sum u_n$ and $\sum v_n$ be summable $|A|$ whenever $\sum|u_n|<\infty$ is that $\sum v_n\epsilon\{A^*\}$. Let A, B, C be triangular matrices and let the first two have inverses. In terms of these matrices and their inverses, conditions are given which imply that the Cauchy product of $\sum u_n$ and $\sum v_n$ is summable $|C|$ whenever $\sum u_n$ is summable $|A|$ and $\sum v_n$ is summable $|B|$. Applications to Riesz matrices (R, q_n) are given. The proofs are based on work of Miss Mears [Ann. of Math. (2) 38, 594–601 (1937)].
R. P. Agnew (Ithaca, N. Y.).

Forsythe, G. E. and Schaeffer, A. C. Remarks on regularity of methods of summation. Bull. Amer. Math. Soc. 48, 863–865 (1942). [MF 7501]

The following is a generalization of a result obtained by Tamarkin [Bull. Amer. Math. Soc. **41**, 241–243 (1935)] by use of a theorem of Banach. If the matrix a_{mn} is such that to each sequence x_n converging to 0 there corresponds an index m_0 such that $\sum a_{mn}x_n$ has bounded partial sums when $m>m_0$, then there is an index m_1 such that $\sum_{n=1}^{\infty}|a_{mn}|<\infty$ when $m>m_1$. The paper under review gives an elementary proof of this theorem, following the method used by the reviewer [Bull. Amer. Math. Soc. **45**, 689–730 (1939), especially, pp. 709–710; these Rev. **1**, 50] in establishing the result for kernel transformations and hence also for sequence-to-function and matrix transformations.
R. P. Agnew (Ithaca, N. Y.).

San Juan, Ricardo. An algorithm for summation of divergent series. Union Mat. Argentina, Publ. no. 21, 6 pp. (1941). (Spanish) [MF 7477]

The method of summability, by which $\sum a_nz^n$ is summable to s if

$$s=\lim_{\alpha\to 1-}\int_0^{\infty}e^{-t}\sum_{n=0}^{\infty}(t^{\alpha n}/n!)a_nz^n dt,$$

and similar methods of summability are discussed briefly.
R. P. Agnew (Ithaca, N. Y.).

Garabedian, H. L. The Cesàro kernel transformation. Amer. Math. Monthly 49, 296–301 (1942). [MF 6691]

For each positive number α, there is given a class of Volterra integral transformations which include the Cesàro integral transformation C_α. The class contains C_β when $\beta>\alpha$. *R. P. Agnew* (Ithaca, N. Y.).

Garabedian, H. L. Hausdorff integral transformations. Ann. of Math. (2) 43, 501–509 (1942). [MF 7004]

Let $(H, \varphi(x))$ denote the integral transformation

$$(H, \varphi(x))\qquad v(x)=\int_0^{\infty}u(y)d\varphi(y/x)$$

from $u(x)$ to $v(x)$. Assuming that φ is normalized by the condition $\varphi(0)=0$, the conditions (1) $\varphi(x)$ has bounded variation over $0\leqq x\leqq 1$, (2) $\varphi(x)$ is continuous at $x=0$ and $\varphi(1)=1$ are necessary and sufficient for regularity of $(H, \varphi(x))$ over the class BC of bounded continuous functions. (The condition (3) $\varphi(x)=\frac{1}{2}[\varphi(x-0)+\varphi(x+0)]$, $0<x<1$, may be imposed as another normalizing condition.) Further results apply to mass functions $\varphi(x)$, satisfying the above conditions, continuous over $0\leqq x<1$. If $(H, \varphi_c(x))$ is the product of two transformations $(H, \varphi_a(x))$ and $(H, \varphi_b(x))$, then $\varphi_c(x)$ is given in terms of $\varphi_a(x)$ and $\varphi_b(x)$ by the Silverman-Schmidt equations

$$(4)\qquad \varphi_c(x)=\int_0^1\varphi_a(x/t)d\varphi_b(t),\quad \varphi_c(x)=\int_0^1\varphi_b(x/t)\varphi_a(t).$$

The conditions (4) imply that $(H, \varphi_c(x))\supset(H, \varphi_a(x))$ over the class BC. *R. P. Agnew* (Ithaca, N. Y.).

Greenberg, Herbert J. and Wall, H. S. Hausdorff means included between $(C, 0)$ and $(C, 1)$. Bull. Amer. Math. Soc. 48, 774–783 (1942). [MF 7283]

Let $\phi(u)$ be any function of bounded variation on the interval $0\leqq u\leqq\infty$ such that $\phi(\infty)-\phi(0)=1$. Then it is shown that the function $\alpha(z)=\int_0^{\infty}d\phi(u)/(1+zu)$ is a regular moment function. Thus, the sequence $\{\alpha(n)\}$ is a regular moment sequence defining a regular Hausdorff method of summation $[H, \alpha(n)]$. When $\phi(u)$ is further restricted to be monotone then the inclusion relations $(C, 0)\subset[H, \alpha(n)]\subset(C, 1)$ are obtained. Conditions under which $[H, \alpha(n)]$ is equivalent to $(C, 0)$ or to $(C, 1)$ are obtained which are analogous to the conditions found by Scott and Wall [Trans. Amer. Math. Soc. **51**, 255–279 (1942); these Rev. **3**, 297] for the special case where $\phi(u)\equiv 1$ for $u\geqq 1$, $\phi(0)=0$. In particular, $(C, 0)\approx[H, \alpha(n)]$ if and only if $\phi(u)$ is discontinuous at $u=0$, and $[H, \alpha(n)]\approx(C, 1)$ if and only if $\int_0^{\infty}d\phi(u)/u<\infty$. Finally, certain transformations of moment sequences, in particular the Hausdorff transformation, are discussed. *H. L. Garabedian* (Evanston, Ill.).

Agnew, Ralph Palmer. Analytic extension by Hausdorff methods. Trans. Amer. Math. Soc. 52, 217–237 (1942). [MF 7117]

This paper is occupied primarily with a study of the analytic extension of a power series $\sum c_n z^n$ with the aid of a regular Hausdorff method of summation $H(\chi)$, with the associated mass function $\chi(t)$, $0 \leqq t \leqq 1$. It is first proved that, if $\chi(t)$ is a regular mass function with no removable discontinuities, and if r is the greatest lower bound of all numbers ρ such that $\chi(t)=1$ when $\rho \leqq t \leqq 1$, then (i) $\sum z^n$ is summable $H(\chi)$ to $1/(1-z)$ at each point z interior to the circle $|z-(1-r^{-1})\zeta|=r^{-1}$, and (ii) $\sum z^n$ is nonsummable $H(\chi)$ at each point exterior to the same circle. Part (i) of this theorem, by far the simpler part, was proved by Garabedian and Wall [Northwestern University Studies in Mathematics and the Physical Sciences, no. 1, Evanston, Ill., 1941, pp. 87–132; these Rev. **3**, 297]. Now, corresponding to each vertex ζ of the Mittag-Leffler star, let $B(r, \zeta)$ denote the set of points z for which $|z-(1-r^{-1})\zeta| < r^{-1}|\zeta|$, and let $B(r)$ denote the set of inner points of the intersection of the sets $B(r, \zeta)$. It is then proved that $\sum c_n z^n$, a power series with a positive finite radius of convergence, is uniformly summable $H(\chi)$ over each closed subset of $B(r)$.

Collective Hausdorff summability $\mathfrak{H}$ is defined, and some properties of the method $\mathfrak{H}$ are obtained. A series $\sum u_n$ is said to be summable $\mathfrak{H}$ to the value σ if it is summable to the value σ by at least one regular method $H(\chi)$. The following Tauberian gap theorem is proved. If $n_0, n_1, n_2, \cdots$ is a sequence of integers for which $0=n_0<n_1<n_2<\cdots$ and $n_{p+1}/n_p \to \infty$ as $p\to\infty$, if $\sum u_n$ is a series for which $u_n=0$, $n \neq n_0, n_1, n_2, \cdots$, and if $\sum u_n$ is summable $\mathfrak{H}$, then $\sum u_n$ is convergent.

The last section of the paper involves results obtained by consideration of the zeros of the moment function $\mu(z)=\int_0^1 t^z \chi(t)$, associated with a regular mass function $\chi(t)$. A similar technique has been used extensively in a recent paper by W. W. Rogosinski [Proc. Cambridge Philos. Soc. **38**, 166–192 (1942); these Rev. **3**, 296].

H. L. Garabedian (Evanston, Ill.).

Leighton, Walter and Thron, W. J. On value regions of continued fractions. Bull. Amer. Math. Soc. **48**, 917–920 (1942). [MF 7512]

The note gives the value-region for the approximants to the continued fraction

$$1+\frac{a_1|}{|1}+\frac{a_2|}{|1}+\cdots,$$

assuming the elements a_i lie in the parabola

$$\rho \leqq 2d(1-d)/(1-\cos\theta), \qquad \tfrac{1}{2} \leqq d < 1.$$

This is an extension of a particular result ($d=\frac{1}{2}$) due to Scott and Wall. *J. A. Shohat* (Philadelphia, Pa.).

Bankier, J. D. and Leighton, Walter. Numerical continued fractions. Amer. J. Math. **64**, 653–668 (1942). [MF 7169]

A discussion is given of the periodic continued fraction

$$(1) \qquad b_0+\frac{a_1|}{|b_1}+\cdots+\frac{a_{k-1}|}{|b_{k-1}}+\frac{a_k|}{|b_0}+\frac{a_1|}{|b_1}+\cdots,$$

a_n, b_n complex, $a_n \neq 0$, mainly concerning the nature of the number it represents. The authors also discuss the relation of (1) to

$$(2) \qquad \frac{a_k|}{|b_{k-1}}+\frac{a_{k-1}|}{|b_{k-2}}+\cdots+\frac{a_2|}{|b_1}+\frac{a_1|}{|b_0}+\frac{a_k|}{|b_{k-1}}+\cdots,$$

and the special case of "proper" continued fractions (a_n, b_n integers properly restricted). Results are obtained generalizing classical ones (for "regular" continued fractions). The proofs proceed along well established lines.

J. Shohat (Philadelphia, Pa.).

Bradshaw, J. W. Modified continued fractions. Amer. Math. Monthly **49**, 513–519 (1942). [MF 7341]

The author applies his method for increasing the rapidity of convergence of sequences to the convergents of a continued fraction. If the last partial quotient of the nth convergent is a rational function of n, this function is replaced by another rational function which does not affect the limit but increases the rapidity of convergence of the continued fraction. The method is illustrated on Euler's continued fraction for $\pi/2$. *P. W. Ketchum.*

Special Functions

Silberstein, Ludwik. Differentially cyclical sets of functions. An extension of the concept of hyperbolic functions. Philos. Mag. (7) **33**, 457–461 (1942). [MF 7071]

A relation between n functions $f_1, f_2, \cdots, f_n$ which satisfy $f_1'=f_2$, $f_2'=f_3$, $\cdots$, $f_n'=f_1$ is expressed by the statement that the Jacobian of $f_1, f_n, f_{n-1}, \cdots, f_2$ is unity. A method of finding addition theorems is illustrated by the case $n=3$ when the relations are

$$f(x+y)=f(x)f(y)+g(x)h(y)+h(x)g(y),$$

$$g(x+y)=h(x)h(y)+f(x)g(y)+g(x)f(y),$$

$$h(x+y)=g(x)g(y)+h(x)f(y)+f(x)h(y).$$

Expressions are then found for $f(-x)$, $g(-x)$, $h(-x)$. In particular, $f(-x)=f^2(x)-g(x)h(x)$. *H. Bateman.*

Banerjee, D. P. On the application of the operational calculus to the expansion of a function in a series of Legendre's functions of the second kind. Proc. Edinburgh Math. Soc. (2) **7**, 1–2 (1942). [MF 7488]

An integral function $\varphi(\omega)$ can be expanded as a series $\sum a_n i^{n+1} Q_n(i\omega)$, where the coefficients a_n are determined by expanding the Laplace transform of φ in a series of Bessel functions of order $n+\frac{1}{2}$. *M. C. Gray.*

Humbert, Pierre. Sur les fonctions K de Bessel. Mathematica, Timişoara **17**, 59–64 (1941). [MF 6731]

Employing Heaviside's calculus several formulae are derived concerning Bessel functions and Laplace transforms.

O. Szász (Cincinnati, Ohio).

Shanker, Hari. On the expansion of the product of two parabolic cylinder functions of non integral order. Proc. Benares Math. Soc. (N.S.) **2**, 61–68 (1940). [MF 6649]

A known expansion for the product of two parabolic cylinder functions of integral orders n and m as a series of parabolic cylinder functions is extended to non-integral values of n, m. An expansion for products of the form $D_n(iz)D_m(z)$ (n arbitrary, m integral) is also obtained, and some integrals involving products of the D_n functions are evaluated. *M. C. Gray* (New York, N. Y.).

Sharma, J. L. On the recurrence formulae of the generalized Lamé functions. Proc. Benares Math. Soc. (N.S.) 2, 43–51 (1940). [MF 6647]

Halphen [Fonctions elliptiques, Gauthier-Villers, Paris, 1888, t. 2, pp. 494–502] has shown that Lamé's equation

$$d^2y/du^2=[n(n+1)\wp(u)+B]y$$

has the solutions $F_n(u)$ and $F_n(-u)$, where

$$F_n(u)=\prod_{r=1}^{n}\left[\frac{\sigma(u+a_r)}{\sigma(u)\sigma(a_r)}e^{-u\zeta(a_r)}\right].$$

These solutions are doubly periodic functions of the second kind. By the use of classical theorems [Liouville, Hermite] in the theory of doubly periodic functions, the author shows in a simple and direct manner that the functions $F_n(u)$ satisfy recurrence relations of the type:

$$\phi_r(u)F_n(u)+\delta\phi_{r+1}(u)F_{n-1}(u)-\epsilon\phi_{r+2}(u)F_{n-2}(u)=0,\qquad n \text{ odd};\ r=\tfrac{1}{2}(n-1),$$

$$\phi_r(u)F_n(u)+\delta\phi_{r+1}(u)F_{n-1}(u)-\epsilon\phi_{r+3}(u)F_{n-3}(u)=0,\qquad n \text{ even};\ r=\tfrac{1}{2}(n-2),$$

where

$$\phi_r(u)=\frac{1}{[\sigma(u)]^r}\prod_{j=1}^{r}\frac{\sigma(u+\alpha_j)}{\sigma(\alpha_j)},$$

$$\phi_{r+1}(u)=\frac{1}{[\sigma(u)]^{r+1}}\prod_{j=1}^{r+1}\frac{\sigma(u+\beta_j)}{\sigma(\beta_j)},$$

$$\phi_{r+2}(u)=\frac{1}{[\sigma(u)]^{r+2}}\prod_{j=1}^{r+2}\frac{\sigma(u+\gamma_j)}{\sigma(\gamma_j)},$$

and δ, ϵ, α_j, β_j, γ_j are suitably determined constants. Other analogous recurrences satisfied by the $F_n(u)$ are also indicated. [See also a paper by the same author in Philos. Mag. (7) 22, 1123–1129 (1936).] *M. A. Basoco.*

Erdélyi, A. Integral equations for Lamé functions. Proc. Edinburgh Math. Soc. (2) 7, 3–15 (1942). [MF 7489]

The principal theorem established in this paper is that the integral equation

$$E_n^s(x)=\lambda_s\int_{-2k}^{2k}Y_n(\theta,\varphi)E_n^s(y)dy,$$

where $Y_n(\theta,\varphi)$ is any surface harmonic of degree n, is the most general equation satisfied by the $2n+1$ linearly independent mutually orthogonal and normalized Lamé polynomials of degree n, $E_n^s(x)$ $(s=0,1,\cdots,2n)$. Integral equations for transcendental Lamé functions of integral degree n and for products of Lamé polynomials are also obtained. Finally the author discusses the corresponding results for Mathieu functions regarded as limiting cases of the Lamé polynomials. *M. C. Gray* (New York, N. Y.).

Varma, R. S. Some infinite integrals involving Whittaker functions. Proc. Benares Math. Soc. (N.S.) 2, 81–84 (1940). [MF 6651]

Discussion of the integral

$$\int_0^\infty x^{l-1}e^{-x/2}W_{k,m}(x)\ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\cdots,\alpha_p\\ \beta_1,\beta_2,\cdots,\beta_q\end{matrix};-x^2t^2\right]dx,$$

which can be evaluated in terms of a generalized hypergeometric function ${}_{p+4}F_{q+2}$, for special values of the various parameters. *M. C. Gray* (New York, N. Y.).

Daum, J. A. The basic analogue of Kummer's theorem. Bull. Amer. Math. Soc. 48, 711–713 (1942). [MF 7269]

A basic series analogue of Kummer's theorem on the sum of a ${}_2F_1$ of argument -1 is given as

$${}_2\Phi_1\left[\begin{matrix}a,b\\ aq/b\end{matrix};-q/b\right]=\frac{\Omega(aq/b)\Omega(qa^{\frac{1}{2}})\Omega(-qa^{\frac{1}{2}})\Omega(-q/b)}{\Omega(aq)\Omega(-q)\Omega(qa^{\frac{1}{2}}/b)\Omega(-qa^{\frac{1}{2}}/b)},$$

where

$$\Omega(z)=\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-zq^n}.$$

From this another formula is obtained giving a relation between a modified type of basic series and a ${}_3\Phi_2$, but the author points out that it cannot be regarded as the basic series analogue of Dixon's theorem on the sum of a well-poised ${}_3F_2$ with unit argument. *M. C. Gray.*

Mohan, B. Infinite integrals involving Struve's functions. Quart. J. Math., Oxford Ser. 13, 40–47 (1942). [MF 7448]

The author is concerned with the evaluation of certain infinite integrals involving Struve's functions, which are defined by

$$H_\nu(z)=\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+\frac{3}{2})\Gamma(\nu+r+\frac{3}{2})}$$

[see G. N. Watson, Theory of Bessel Functions, Cambridge University Press, 1922, §10.4]. The following is a typical result:

$$\int_0^\infty x^{p-1}e^{-ax^2}H_\nu(bx)dx=\frac{b^{\nu+1}\Gamma(\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2})}{2^{\nu+1}\pi^{\frac{1}{2}}\Gamma(\nu+\frac{3}{2})a^{\frac{1}{2}\nu+\frac{1}{2}p+\frac{1}{2}}}\cdot{}_2F_2\left(\begin{matrix}1,\ \nu+p+\frac{1}{2}\\ \frac{3}{2},\ \nu+\frac{3}{2}\end{matrix};-\frac{b^2}{4a}\right)$$

$(a, b>0;\ \nu, m, n, p \text{ real})$

which is obtained by substituting the series for $H_\nu(x)$ in the integral and inverting the order of summation and integration. Forty seven such integrals (some of which are special cases of the preceding result) are listed.

M. A. Basoco (Lincoln, Neb.).

Mohan, B. Properties of a certain confluent hypergeometric function. Bull. Calcutta Math. Soc. 33, 99–103 (1941). [MF 7025]

The author is concerned with the function

$$\psi_\nu(x)=\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}+k)x^{\nu+2k}}{2^{\nu+2k}k!\Gamma(1+\nu+k)},\qquad \Re(\nu)>-1,$$

which is absolutely and uniformly convergent in any finite interval. In a familiar notation, this function may also be written

$$\psi_\nu(x)=\frac{(x/2)^\nu\pi^{\frac{1}{2}}}{\Gamma(\nu+1)}\,{}_1F_1(\tfrac{1}{2},\nu+1;x^2/4).$$

No indication is given as to the origin of this function; some of the properties listed are:

$$\psi_\nu'(x)=\psi_{\nu-1}(x)-(\nu/x)\psi_\nu(x),$$

$$(x/2+\nu/2)\psi_\nu(x)=\psi_\nu'(x)+(\nu+\tfrac{1}{2})\psi_{\nu+1}(x),$$

$$(x^{-1}d/dx)^m(x^\nu\psi_\nu(x))=x^{\nu-m}\psi_{\nu-m}(x),\qquad m>0 \text{ an integer},$$

$$\psi_\nu(x)=\frac{(x/2)^\nu}{\Gamma(\frac{1}{2}+\nu)}\int_0^1\frac{(1-u)^{\nu-\frac{1}{2}}}{u^{\frac{1}{2}}}e^{ux^2/4}du,$$

$$\psi_\nu(x)\sim x^{-\nu-1}e^{x^2/4},\qquad \text{for } x \text{ large}.$$

The function $\psi_\nu(x)$ is shown to satisfy the differential equation

$$\frac{d^2y}{dx^2}+\left(\frac{2-x^2}{2x}\right)\frac{dy}{dx}+\left(\frac{\nu}{2}-\frac{1}{2}-\frac{\nu^2}{x^2}\right)y=0.$$

Several integrals involving $\psi_\nu(x)$ are evaluated in terms of the hypergeometric function. *M. A. Basoco.*

MacRobert, T. M. Some integrals involving E-functions and confluent hypergeometric functions. Quart. J. Math., Oxford Ser. 13, 65–68 (1942). [MF 7630]

The E-function used in this paper may be defined by the integral

$$E(\alpha,\beta:x)=\Gamma(\alpha)\int_0^\infty e^{-\lambda}\lambda^{\beta-1}\left(1+\frac{\lambda}{x}\right)^{\alpha}d\lambda,\quad \Re(\beta)>0;$$

thus it represents a modified form of the $W_{k,m}$ function. Infinite integrals involving either one or two E-functions are evaluated, and the corresponding integrals for the $W_{k,m}$ functions are included. *M. C. Gray* (New York, N. Y.).

Bateman, H. An orthogonal property of the hypergeometric polynomial. Proc. Nat. Acad. Sci. U.S.A. **28**, 374–377 (1942). (MF 7186]

The author studies the relation of various orthogonal polynomials to the hypergeometric function. The relation

$$\operatorname{sech} x=\sum_{n=-\infty}^{\infty}\frac{\sin \pi(x-n)}{\pi(x-n)}\operatorname{sech} n$$

(which the author refutes by numerical calculations) is obviously impossible since the right-hand series represents an entire function of x whereas sech x has poles.

G. Szegö (Stanford University, Calif.).

Bateman, H. Some asymptotic relations. Proc. Nat. Acad. Sci. U.S.A. **28**, 371–374 (1942). [MF 7185]

The integral $\int_0^\infty e^{-xa}\operatorname{sech} a \tanh^n a\, da$ is expressed in terms of the hypergeometric function. By means of this representation a simple asymptotic formula is obtained as $n\to\infty$. A more general integral is also dealt with in a similar manner. *G. Szegö* (Stanford University, Calif.).

Integral Equations

Pitt, H. R. General Mercerian theorems. II. Proc. London Math. Soc. (2) **47**, 248–267 (1942). [MF 6595]

In an earlier paper [Proc. Cambridge Philos. Soc. **34**, 510–520 (1938)] the author gave conditions under which

$$g(x)=\int_{-\infty}^{\infty}f(x-y)dk(y), \tag{1}$$

with $k(y)$ of bounded variation in $(-\infty,\infty)$ and $f(x)$ bounded and continuous, has a solution

$$f(x)=\int_{-\infty}^{\infty}g(x-y)dp(y), \tag{2}$$

with $p(y)$ of bounded variation. (It then follows that $f(x)\to 0$ as $x\to\infty$ if $g(x)\to 0$, and this is a Mercerian theorem.) The essential condition was that

$$K(\omega)=\int_{-\infty}^{\infty}e^{-\omega x}dk(x) \tag{3}$$

should have a bounded reciprocal in a strip $0\leqq\sigma\leqq\sigma_1$ of the $\omega=\sigma+it$ plane. Here the author extends his results to cases where $[K(\omega)]^{-1}$ is not bounded. (I) Hypothesis: $\sigma_2>\sigma_1>0$; $k(y)\epsilon V^*$ (functions of bounded variation with no singular component); (3) converges absolutely in $0\leqq\sigma\leqq\sigma_2$; $K(\omega)$ has zeros of order a_n at a finite set of points ω_n in $0<\sigma\leqq\sigma_1$; $[K(\omega)]^{-1}$ is bounded in $0\leqq\sigma\leqq\sigma_1$ except in neighborhoods of ω_n; $f(x)$ is continuous and $f(x)e^{-\sigma x}$ is bounded for each fixed σ in $\sigma_1<\sigma\leqq\sigma_2$; $g(x)$ defined by (1) is bounded. Conclusion:

$$f(x)=\int_{-\infty}^{\infty}g(x-y)dp(y)+\sum_{n=1}^{N}\sum_{\nu=1}^{a_n}A_{n\nu}e^{\omega_n x}x^{\nu-1},$$

where $p(x)\epsilon V^*$ and $A_{n\nu}$ are constants. If also

$$\varliminf_{T\to\infty}\ \varliminf_{x\to\infty}\left|(2T)^{-1}\int_{x-T}^{x+T}f(y)e^{-\omega_n y}dy\right|=0$$

for all n, (2) is true with $p(x)\epsilon V^*$. (II) In case $\sigma_1=\sigma_2=0$ and $[K(it)]^{-1}$ is bounded outside a finite set of intervals $(a_n-\epsilon, b_n+\epsilon)$, under a rather complicated set of conditions involving results from the author's theory of general harmonic analysis [Proc. London Math. Soc. (2) **46**, 1–18 (1940); these Rev. **1**, 139] it follows that

$$\limsup_{x\to\infty}|f(x)|\leqq C\limsup_{x\to\infty}|g(x)|.$$

(III) The case where $K(\omega)$ has zeros both on $\sigma=0$ and in the strip $0<\sigma\leqq\sigma_1$ is also considered. *R. P. Boas, Jr.*

Collatz, L. Einschliessungssatz für die Eigenwerte von Integralgleichungen. Math. Z. **47**, 395–398 (1941). [MF 6725]

Let $K(s,t)$ be a symmetric kernel subject to the usual continuity conditions. Let $F_0(t)$ be continuous,

$$F_1(s)=\int_a^b K(s,t)F_0(t)dt$$

and suppose that $\phi(s)=F_0(s)/F_1(s)$ satisfies the inequality $0\leqq m\leqq\phi(s)\leqq M$ in (a,b); then there exists at least one characteristic value λ of $F(s)=\lambda\int_a^b K(s,t)F(t)dt$ lying between m and M. The proof is based on the remark that $G(s)=F_0(s)$ is a characteristic function of the integral equation $G(s)/\phi(s)=\lambda^*\int_a^b K(s,t)G(t)dt$; it belongs to $\lambda^*=1$. The theorem now follows from classical maximum-minimum properties of characteristic values. [For similar theorems by the same author, cf. Math. Z. **46**, 692–708 (1940); these Rev. **2**, 312.] *W. Feller* (Providence, R. I.).

Vecoua, N. Integralgleichungen vom Fredholmschen Typus mit Integralen im Hadamardschen Sinne. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 7, 113–146 (1940). (Russian. German summary) [MF 5306]

The author considers the equation

$$\varphi(t)-\lambda\int_a^b(b-\tau)^{-p-\mu}k(t,\tau)\varphi(\tau)d\tau=f(t),$$

where in the integral the "finite" part, only, is involved (in the sense of Hadamard); integer $p>0$; $0<\mu<1$; $k(t,\tau)$ has partial derivatives of order $2p$, continuous for $a\leqq t$, $\tau\leqq b$; f has a continuous pth order derivative. By a method of successive approximations a solution for $|\lambda|$ small is constructed. Proof of uniqueness of the solution is based on a certain function-space theorem of Caccioppoli-Banach. For $|\lambda|$ small a theory of resolvent kernels is developed. The author then constructs solutions for all λ. Finally, it is shown that all the theorems of a paper by Riesz [Acta

Math. 41, 81–98 (1918)] hold for the function space M involved (here M is the space of all continuous functions, defined on an interval $[a, b]$ and therein possessing continuous first order derivatives). *W. J. Trjitzinsky.*

Vecoua, I. On a class of singular integral equations and some boundary value problems of potential theory. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 10, 73–92 (1941). (Georgian. Russian summary) [MF 7385]

As is known, singular integral equations (where the integrals are taken in the sense of Cauchy principal value) in general do not have bounded solutions. We give a method of determining bounded solutions and apply the results obtained to the solution of the Dirichlet problem for a plane with slits. *From the Russian summary by the author.*

Hildebrand, F. B. Note on the integro-differential equation of a problem in the theory of plane stress. J. Math. Phys. Mass. Inst. Tech. 21, 19–22 (1942). [MF 6776]

Some conjectures of E. Reissner [Proc. Nat. Acad. Sci. U.S.A. 26, 300–305 (1940); these Rev. 2, 32] regarding the character of the solution of a certain singular integral equation are shown to be correct. *J. J. Stoker.*

de Moraes, Abrahão and Schönberg, Mario. On the equation of dielectric media. Anais Acad. Brasil. Ci. 12, 137–153 (1940). (Portuguese) [MF 6573]

The integro-differential equation

$$aU'(t)+bU(t)+\int_0^t \phi(t-\tau)U'(\tau)d\tau+\psi(t)=0$$

is reduced by a simple integration with respect to t to the Volterra equation

$$U(x)=F(x)+\int_0^t K(x-t)U(t)dt,$$

where $K(x)=-(b+\phi(x))/a$; the latter equation is solved in the usual fashion using Laplace integrals. A similar method works also if $aU'(t)$ is replaced by $\sum_{k=n}^{n} a_k U^{(k)}(t)$.
W. Feller (Providence, R. I.).

Cameron, R. H. Quadratic convolution equations. J. Math. Phys. Mass. Inst. Tech. 21, 57–62 (1942). [MF 7246]

A discussion of equations of the form

$$(1)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x-t-u)dF(u)dA(t) +2\int_{-\infty}^{\infty} F(x-t)dB(t)+C(x)=0,$$

where $A(x)$, $B(x)$, $C(x)$ are right continuous functions of bounded variation on $-\infty<x<\infty$ and solutions of the same type are to be found. Let

$$D(x)=\int_{-\infty}^{\infty}[B(x-t)dB(t)-A(x-t)dC(t)].$$

In the simplest theorem, the author supposes that the Fourier-Stieltjes transforms of $A(x)$ and of $D(x)$ are bounded away from zero and, in addition, that $A(x)$, $B(x)$, $C(x)$ are real non-singular and that $C(-\infty)=0$. Let $d_1=D(+\infty)-D(-\infty)$ and d_2 be the algebraic sum of the jumps of $D(x)$. If d_1 and d_2 have the same sign, then (1) has two and only two right continuous solutions which vanish at $-\infty$, and both solutions are non-singular functions of bounded variation. The solutions are real if d_1 and d_2 are positive, and conjugate complex if d_1 and d_2 are negative. If d_1 and d_2 have opposite signs, there is no solution of bounded variation. If $A(x)$, $B(x)$, $C(x)$ are complex, but the other assumptions are as before, the ratio of the Fourier-Stieltjes transform of the discrete part of $D(x)$ to the transform of $D(x)$ winds about the origin an integral number of times (say n) when t varies from $-\infty$ to ∞. If n is odd, (1) has no solutions of bounded variation, while if n is even there are exactly two solutions, vanishing at $-\infty$, which are of bounded variation and also non-singular. The assumption that the given functions be non-singular may be weakened somewhat. It is essential only to restrict the size of the singular parts of $A(x)$ and $D(x)$. Some special cases are of interest. If $A(x)$ and $B(x)$ are absolutely continuous except for a single jump at the origin and $C(x)$ is absolutely continuous, then the author can specify simple conditions under which (1) will have a single absolutely continuous solution. The case in which the given functions are step functions with jumps at the integers leads to an infinite system of quadratic equations in infinitely many unknowns and the associated Fourier-Stieltjes transforms become ordinary Fourier series. The analysis is largely based upon earlier results of Cameron and Wiener [Trans. Amer. Math. Soc. 46, 97–109 (1939); these Rev. 1, 13] and Wiener and Pitt [Duke Math. J. 4, 420–436 (1938)]. *E. Hille.*

Schönberg, Mario. Fundamentals of a theory of Green's functions. Anais Acad. Brasil. Ci. 13, 85–96 (1941). (Portuguese) [MF 6582]

The author considers a nonhomogeneous linear functional equation $Af=F(P)$, where $F(P)$ is a given function and Af is a linear transformation whose domain and range are classes of functions of P defined for P in a region D of n-dimensional space. The fundamental functional properties of elementary solutions and a Green's function for such an equation are obtained. Generalized elementary solutions and generalized Green's functions are also considered. Finally, these results are applied to the characteristic value problem for the homogeneous functional equation $Af=\lambda f(P)$.
W. T. Reid (Chicago, Ill.).

TOPOLOGY

★**Lefschetz, Solomon. Algebraic Topology.** American Mathematical Society Colloquium Publications, v. 27. American Mathematical Society, New York, 1942. vi+389 pp. $6.00.

This is the fifth book to appear presenting the side of topology centering around homology theory. The others are: O. Veblen, Analysis Situs (V), S. Lefschetz, Topology (L I), H. Seifert and W. Threlfall, Lehrbuch der Topologie (ST) and P. Alexandroff and H. Hopf, Topologie I (AH). (A recent elementary book by M. G. A. Newmann, Cambridge, 1939 and two works [Braunschweig, 1932 and Leipzig, 1938] by K. Reidemeister, should also be mentioned.) The present book (L II) was conceived as a second edition of L I, but actually is almost wholly new, giving in very general and abstract form the modern theories of algebraic (=combinatorial) topology. The place of these five works is somewhat as follows. The first book, V (1922; second edition, 1931) had an enormous influence in increasing interest in

the subject, and was the standard text for a decade. Next (1930) came L I, going far beyond V in material and with a very geometric point of view, but quite difficult to read. In 1934, the appearance of ST gave the young student an excellent first text in the field. In one subject, the fundamental group (with applications), it fills a large gap in earlier works. Soon after (1935), AH was published. It is a very full treatment, considering most of the subjects in L I, and various others, especially some elementary applications. Both ST and AH are written in a clear, detailed style.

At just this time, combinatorial topology took a sudden and rather unexpected leap forward, in the discovery of cohomology theory, largely growing out of Pontrjagin's work. There was therefore no longer a complete text on the subject, and only now has L II filled the gap. Like L I, it seems in advance of its time; unlike L I, it is written with full accuracy and with clarity, but it has also largely lost touch with geometry. The preface states: "Owing to limitations of space and time it has not been possible to take up the applications of algebraic topology." However, some are given in the last chapter, and more appear in Lefschetz's Topics in Topology [see the following review]. For the mature student and worker in the field, L II gives far reaching results, in very abstract style, from quite a unified point of view; the reader must largely furnish himself the appropriate geometric interpretations. An elementary student would find it easier reading in detail than L I, but would get less feeling of what the subject was all about. Most striking of all, there is not a single figure. In style it is quite concise. In each proof the necessary elements are brought in, but sometimes in a somewhat displaced order, making the logical structure of the proof more difficult to ascertain. The modern nature of the book is indicated by the fact that almost all references are to books or papers appearing in 1930 or later. In fact, no attempt is made to give the source of the principal ideas, but only the most recent treatments of them. The general plan of the book is as follows. After introductory chapters on general topology and group theory, the standard homology theory of complexes is taken up, followed by a study of the Alexander duality theory, products, chain-mappings, subdivision, intersections, fixed elements under mappings, and combinatorial manifolds. Next comes a chapter on "nets," "webs," etc., which are then applied to the study of homologies in topological spaces. There follow a final chapter on polyhedra, and two appendices.

We now outline the chapters. I. Standard material on topological spaces, coverings, connectedness, compact spaces, inverse mapping systems and metrization is given (compare AH, chapters I and II). II. Additive groups (written largely by C. Chevalley). A full treatment of Abelian topological groups is presented, followed by mapping systems, the Pontrjagin duality theory (omitting only the proof of the existence of a complete set of characters of a group), and a study of vector spaces over a field. (Some examples would greatly elucidate the last topic.) III. Complexes. The (abstract) complexes of Tucker are used. Throughout the book, both a complex X and its dual X^*, both being either finite or infinite, are treated. (It would be a slight logical simplification to define chains in X^* as linear functions on the integral chains of X, with values in a group.) The "dissection" of a complex into closed and complementary open complexes give the earlier "relative" theory. Duality in the sense of Pontrjagin and of Alexander is studied, and finally "augmentable" and "simple" complexes are treated. (The definition of augmentability is useful only if $\nu_0=\sum x_0^i$ (see (III, 47.1)). IV. The homology groups of a product of complexes are determined. The theory of set- and chain-mappings (Tucker and Lefschetz), generalizing simplicial mappings, is thoroughly studied. It will be one of the principal tools in the rest of the book. Adding certain properties, the theory of subdivision, in the abstract sense, is obtained. V. Multiplications of a pair X, Y into Z, satisfying the single condition $F(\xi^p\times\eta^q) = F\xi^p\times\eta^q+(-1)^p\xi^p\times F\eta^q$, are first presented. The result of interchanging complexes and changing to duals is studied, necessitating a rather laborious discussion. Adding local conditions gives the standard intersections among cocycles and cycles; these are fully treated. In one important respect there is insufficient generality. The theory does not cover the case of intersections when, for example, reals mod 1 are used as coefficients for all cycles, and integers for all cocycles. The algebraic fixed point formula is obtained, and is followed by an extensive study of combinatorial manifolds.

Chapter VI gives the combinatorial theory to be used in the study of the Čech homology theory of general spaces. A net is a directed set of complexes $\{X_\lambda\}$ such that at least one projection (chain-mapping) $\pi_\mu^\lambda : X_\lambda\to X_\mu$ exists if $\lambda>\mu$, the product of two projections is another, and any two π_μ^λ, $\pi_\mu'^\lambda$ carry a cycle into homologous cycles. If the π_μ^λ are unique, we have a spectrum. This abstracts from the original projection spectra of Alexandroff. Changing $>$ to $<$ gives conets and cospectra. A web is a combinatorial analogue of the set of open (or closed) sets of a space with their inclusion relations. Definition (22.4) defines one kind of H-net (homology net), for example, as a system of dual categories of a direct web of subnets of a given net, satisfying certain conditions. The chapter is not easy reading. In VII, the coverings of a space by finite sets of open (or closed) sets are considered; with the corresponding webs and nets, a homology theory arises. The effect of demanding that the space be normal, compact, etc., and the relations to connectedness, are discussed. Cycles "through" and "around" a point are treated. The Mayer-Vietoris formula is generalized to spaces. The theories of Vietoris, Alexander and Kolmogoroff are brought into the situation. (The treatment of Alexander's gratings does not follow his paper, but earlier Proceedings notes.)

The final chapter VIII has a more classical leaning. Euclidean complexes, deformations, dimension, duality, singular chains and the like are treated. The fixed point formula is proved for "quasi-complexes." Some theorems on differentiable manifolds are stated without proof. The main part of the book ends with a beautiful application of fundamental methods: a proof of the recent theorem of H. Hopf that any group manifold (in fact, much more general complexes) has the same homology groups as a product of odd-dimensional spheres. There is a short appendix by Eilenberg and MacLane on the homology groups of infinite complexes and compacts, and a longer one by P. A. Smith on the fixed points of periodic transformations.

There are practically no typographical errors. Some errors in statement are: (I, 8.1) (a mapping must be single-valued), (IV, 27.3), (IV, 29.5); in proof: (I, 12.5), (II, 28.2c), (III, 47.5), (VIII, 23.5). Some proofs which will cause the reader difficulty are: (II, 30), (III, 23.3), (IV, 10.11), (IV, 22.1), (V, 8.9), (V, 29.10b), (VI, 7.2), (VI, 15.2), (VII, 27), (VIII, 19.6), (VIII, 23.7), (VIII, 27.10). The statements (IV, 17.10) and (VI, 3.12) need revision; in (VIII, 23.7), the hypothesis may be omitted. Throughout,

a symbol commonly means either any element or a fixed element of a class, often indiscriminately; thus, γ^p is a fixed cycle or any cycle. This tends to conciseness rather than to clarity. The two symbols γ_i^p, γ_p^i mean different kinds of things; thus γ_2^2 (and t_2^2 in chapter III) is not well defined. As long as one does not apply the theory to particular examples, this causes no great difficulty.

One misses most in the book more discussion of the logical tie-up of the various theories, and examples showing their meaning in simple cases. For example, the fundamental difference between the approaches to the homology theory of a space through Vietoris and singular cycles appears in just one example (VIII, 25.1). To understand the proofs of the standard applications in chapter VIII, one must, on the whole, be quite familiar with chapters VI and VII; thus the applications may be thought of mainly in the light of illustrations of the general theory. The book will certainly be a fundamental source of knowledge for some time to come; it will be interesting to see how well topologists will be able to apply its methods to particular theories. The first half will be well repaying to any topologist; anyone studying the last half will find himself fully rewarded in the end. *H. Whitney* (Cambridge, Mass.).

✱Lefschetz, Solomon. Topics in Topology. Annals of Mathematics Studies, no. 10. Princeton University Press, Princeton, N. J., 1942. vi+137 pp. $2.00.

This book may be considered as a continuation of the author's Algebraic Topology [see the above review]. Written in the same style, it undertakes a further study of geometric and singular complexes, mapping theorems, retraction, and local connectedness. The first chapter is novel in its consideration of infinite polytopes, especially their metrization (theorem of W. Wilson, proof by J. Tukey). The short Chapter II, on singular complexes, considers a set of related notions such as singular cells and singular prismatic cells, singular and continuous complexes, singular and continuous cycles.

The remaining two chapters form the body of the work. Chapter III begins with theorems about "analytic" coverings of a space and imbedding theorems from dimension theory. Borsuk's theory of retraction is now taken up and extended. For example, a complex may be added to any closed set in the Hilbert parallotope P^ω to form a retract of P^ω; a n. a. s. c. for a separable metric space to be an absolute retract [absolute neighborhood retract] is that some topological image A in P^ω be a retract [neighborhood retract] of $I\times P^\omega-(\bar{A}-A)$, I a segment. Chapter IV goes deeply into the relation between various types of local connectedness and retraction properties. Pairs of neighborhoods of a point may be considered such that one, or each singular p-sphere in one, may be contracted through the other to a point; spheres and their deformations may be replaced by cycles and chains bounded by them. As a stronger property, one may assume that mappings of a subcomplex into a space can be extended to the full complex; local conditions may be added. Numerous relations between the resulting definitions are given. For LC^ω compacta, Vietoris and singular homology theories agree. The fixed point formula is extended to LC^ω compacta. As an application, some general fixed point theorems of Schauder are proved. Finally, a brief description of *HLC* spaces and generalized manifolds is given. The book should prove very useful to any worker in this field.

H. Whitney (Cambridge, Mass.).

✱Whyburn, Gordon Thomas. Analytic Topology. American Mathematical Society Colloquium Publications, v. 28. American Mathematical Society, New York, 1942. x+278 pp. $4.75.

To quote the author, "Analytic topology is meant to cover those phases of topology which are being developed advantageously by methods in which continuous transformations play the essential role." In this the author coins a new term to cover the extensive results of the past ten years, obtained largely in this country, on transformation topology. The book falls naturally into two parts: the first six chapters in which the fundamental ideas of topology are developed, and the last six chapters in which the transformation theory proper is studied. The book is almost wholly self-contained, all the necessary ideas of topology which the author uses being developed in the first six chapters, so that the book may be used with no previous knowledge of topology. In fact, the first half of the volume would serve excellently as a textbook for a beginning course in the subject. Only toward the end of the volume are some ideas of combinatorial topology and group theory used without adequate introduction. However, these occur largely in applications, so that the reader who lacks knowledge of these fields loses little of the main development.

Were the volume a mere collection of the theory developed in diverse papers in recent years, it would be worthwhile as a sourcebook for present and future workers in transformations. However, it is much more than that. Some of the results are new and much of the treatment is new, some of the proofs acquiring an elegance and polish they sadly lacked in the original papers. There is an excellent bibliography plus an index giving the location of every definition in the book. The only regret felt by the reviewer was that the author did not take the opportunity to point the road to new developments and possible further extensions of the material.

An idea of the contents of the book may be obtained from the chapter headings and brief comment on their contents. I. Introductory topology. Here are the fundamental definitions and theorems with axioms to give a separable metric space which is used throughout the volume. II. Continuous transformations. Their main properties are developed with the mapping theorems on locally connected continua. III. Cut points. Nonseparated cuttings. The Whyburn theorem that all cut points are of order two except for a countable number, with various generalizations. IV. Cyclic element theory. Development of ideas of cyclic elements and chains; A-sets, and cyclic extensibility and reducibility so useful in analysis of locally connected and semilocally connected spaces. V. Special types of continua. Trees, hereditary locally connected continua, curves. VI. Plane continua. VII. Semicontinuous decompositions and continuous transformations. Includes fundamental properties of nonalternating, monotone and interior transformations. VIII. General properties. Factorization. IX. Applications of monotone and nonalternating transformations. X. Interior transformations. The general theory on the two-dimensional manifold which was begun by Stoïlow but largely developed by Whyburn. XI. Existence theorems. Mappings onto the circle. Conditions for retractions and other mappings onto arcs and circles. XII. Periodicity. Fixed points. Fixed point theorems and properties of various periodic and almost periodic transformations.

W. L. Ayres (Lafayette, Ind.).

Monteiro, António. La notion de fermeture et les axiomes de séparation. Portugaliae Math. 2, 290–298 (1941). [MF 6771]

The separation axioms which appear in the definitions of spaces referred to as regular, normal and completely normal are given equivalent forms in terms of closure.

J. F. Randolph (Ithaca, N. Y.).

Albuquerque, J. La notion de "frontière" en topologie. Portugaliae Math. 2, 280–289 (1941). [MF 6782]

Properties are given which characterize, by means of a primitive notion of frontier of a set, the spaces (F), Hausdorff, regular, normal and completely normal.

J. F. Randolph (Ithaca, N. Y.).

Szymanski, Piotr. La notion des ensembles séparés comme terme primitif de la topologie. Mathematica, Timişoara 17, 65–84 (1941). [MF 6732]

The author proposes three sets of axioms for topology, each based upon the concept of separation as an undefined property. It is shown that each set is equivalent to the axioms of Kuratowski. The fundamental theorems deducible from each axiom are given. The set $\underline{A}$ is defined as the sum of all open sets contained in the set A. The duality between the laws governing this operation and the operation $\overline{A}$ of Kuratowski is shown. *H. M. Gehman* (Buffalo, N. Y.).

Pitcher, Everett and Smiley, M. F. Transitivities of betweenness. Trans. Amer. Math. Soc. 52, 95–114 (1942). [MF 6996]

The authors first make a complete and finished axiomatic study of the triadic relation of betweenness, when four or five points are involved. Their results are summarized in a full-page table, which extends earlier results of Huntington and Kline [Trans. Amer. Math. Soc. 18, 301–325 (1917)]. They consider next the particular "betweenness" relation $(a \cap b) \cup (b \cap c) = b = (a \cup b) \cap (b \cup c)$ in a lattice, suggested by Glivenko, and prove that every lattice satisfies three of their "transitivity" conditions on betweenness, that modularity is equivalent to either of two other such conditions, that distributivity is equivalent to a sixth condition, and implied by a seventh and an eighth, and that being a chain (linearity) is equivalent to any of four others. Thus lattice identities on three elements are shown to be equivalent to betweenness identities on four or five.

G. Birkhoff (Cambridge, Mass.).

Eilenberg, Samuel and Wilder, R. L. Uniform local connectedness and contractibility. Amer. J. Math. 64, 613–622 (1942). [MF 7166]

A study is made of local connectedness and uniform local connectedness in the sense of homotopy, homology and contractibility. The concepts are extended so as to have meaning for a set in relation to a larger containing space; the study is directed toward obtaining relations between a set and its closure. The notions are characterized in various ways and equivalence and implication relations between them are found. In the later sections applications are made to the subsets of Euclidean space. It is shown, for example, that, if a domain complementary to the topological image of an $(n-1)$-sphere in an n-sphere is uniformly locally connected in the homotopy sense for all dimensions 0 to n, the domain is necessarily a singular n-cell. *G. T. Whyburn.*

Begle, Edward G. Locally connected spaces and generalized manifolds. Amer. J. Math. 64, 553–574 (1942). [MF 6949]

In a previous paper [Duke Math. J. 2, 435–442 (1936)] Lefschetz introduced the notions of a realization and a partial realization of a finite complex on a space. This paper gives new definitions of realizations and partial realizations in such a way that the methods developed for LC spaces can be carried over to lc spaces. Vietoris cycles are used throughout and the coefficient group is restricted to be a ring with a unit. Defining local connectivity in terms of Vietoris cycles, the author compares his definition with various others and shows that they are equivalent in the important cases. Several of the results are shown to extend to compact (that is, bicompact normal) spaces. The paper concludes with a number of examples and shows how the results obtained permit a simplification of the usual discussion of generalized manifolds. *D. W. Hall.*

Wallace, A. D. Separation spaces. II. Anais Acad. Brasil. Ci. 14, 203–206 (1942). [MF 7463]

This note continues the study of separation spaces initiated by the author in a previous paper [Ann. of Math. (2) 42, 687–697 (1941); these Rev. 3, 57]. (The axioms which every separation space must satisfy can be found in the original paper and also in the review just cited.) The present note uses the undefined concept $X \mid Y$ ("X is separated from Y") of the original paper to partially order the subsets of the separation space S. A family G of subsets of S is called a nested family provided that if X, Y are elements of G then either $X \mid Y'$ or $Y \mid X'$, where X', Y' denote the complements of X, Y, respectively. Among other things the author shows that if S is a compact topological space and $X \mid Y$ means that $\bar{X}Y + X\bar{Y} = 0$, then any nested family of non-void sets in S has a non-void intersection. In view of this result he calls a separation space compact provided it satisfies this latter condition. With this definition of compactness he is able to prove that every compact connected separation space S containing at least two points must contain at least two non-cutpoints of itself.

D. W. Hall (College Park, Md.).

Sebastião e Silva, J. Les ensembles fermés et le problème de Wiener. Portugaliae Math. 3, 124–131 (1942). [MF 7086]

The author considers a problem on abstract spaces proposed by Wiener [see Fréchet, Les espaces abstraits, Gauthier-Villars, Paris, 1928, pp. 196–201]. Besides establishing anew Wiener's results, he proves that: (1) The class of S spaces is identical with the class of those F spaces dense in themselves. (2) The class of mappings Σ of the space 1 into itself is identical with the class Γ of bicontinuous (in the topology generated by Σ) mappings of 1 into itself if and only if (a) Σ is a group; (b) every group of mappings equivalent to Σ is in Σ; (c) the group Σ is not a proper invariant subgroup of any group of mappings of 1 on itself. *E. R. Lorch* (New York, N. Y.).

Vicente Gonçalves, J. Quelques résultats concernant les régions simples. Portugaliae Math. 2, 247–270 (1941). [MF 6520]

By a (planar, bounded) simple region is understood the interior plus boundary of a Jordan contour. A point x is subject to a set E if every continuous curve connecting x and ∞ contains a point of the frontier of E; in the contrary case, x is free with respect to E. A set is completely subject

to E if all its points are subject to E. A sequence of sets $\cdots$, E_{n-1}, E_n, E_{n+1}, $\cdots$, finite or infinite in one or both directions, is a chain if (α) none of the sets is completely subject to any other, (β) every intermediate set has at least one point in common with its predecessor, and also at least one with its successor, (γ) there is no point common to three sets. A nonintermediate set is an end; in a chain which is infinite to the right, the right end is defined to be the set of limit points of sequences of points x_n belonging respectively to E_n, $n>0$; similarly for the left. A chain is closed if its ends are not disjoint. A radiation is a collection of a finite or denumerable number of bounded continua K_n such that (α) none is completely subject to another, (β) the product of two, when not equal to 0, is either connected or reduces to a point, (γ) two disjoints K_n's are members of a chain of K_n's, (δ) there is no closed chain. The principal result of the present paper is contained in the following theorem, and more particularly, in its extension, under certain conditions, to the case of a denumerable number of E_ν: If the sets E_ν, $\nu=1, 2, \cdots, n$, constitute the radiation R, and F is a closed set all of whose points are free with respect to each E_ν, then R may be enclosed in a simple region containing no point of F. *H. Blumberg.*

Freudenthal, Hans. Simplizialzerlegungen von beschränkter Flachheit. Ann. of Math. (2) **43**, 580–582 (1942). [MF 7012]

If an r-simplex T_r is given with its vertices ordered, a natural cutting up of T_r into 2^r simplexes is given, each similar to a "conjugate" of T_r. Hence any finite complex K has an arbitrarily fine subdivision K', such that

$$\rho(K')=\max_{T_r \epsilon K'}[(\text{diam } T_r)^r/\text{vol } T_r]$$

has a bound $\rho_0(K)$ independent of K'. *H. Whitney.*

Gilbert, Paul W. n-to-one mappings of linear graphs. Duke Math. J. **9**, 475–486 (1942). [MF 7331]

The paper contains an interesting group of theorems on the existence of exactly n-to-one mappings on finite linear graphs. It is shown that an n-to-1 mapping for every odd n may be defined on any linear graph. Further, if the graph has a mapping which is almost 2-to-1, then it has a mapping which is exactly n-to-1 for every $n\neq 2$. We will say that a mapping is almost 2-to-1 if all inverse sets are pairs of points with at most one exception, and this exceptional inverse set is a single point. (The author uses the term "mapping of class I" for almost 2-to-1.) Every linear graph which is a boundary curve has an almost 2-to-1 mapping, and the set of all images of all boundary curve graphs under such mappings is the set of all connected graphs. Finally, it is shown that, if an exactly 2-to-1 mapping exists, the Euler characteristic of the graph is even; if there is an almost 2-to-1 mapping which is not exactly 2-to-1, then the Euler characteristic is odd. Examples are given which show that the converses are not true. *W. L. Ayres.*

Whitehead, George W. On the homotopy groups of spheres and rotation groups. Ann. of Math. (2) **43**, 634–640 (1942). [MF 7396]

This paper is a study of certain questions pertaining to the homotopy groups π_m of spheres S^n. The main tool is the association of a mapping ϕ_f of S^{m+n+1} into S^{n+1} with any mapping f of $S^m\times S^n$ into S^n, generalizing a procedure of H. Hopf for $m=n$ [Fund. Math. **25**, 427–440 (1935)]. Let e_1, e_2, $\cdots$ be orthogonal unit vectors, and let $E_r{}^s$ be the plane of $e_r, \cdots, e_s$. If $p\epsilon E_1^{m+1}$, $q\epsilon E_{m+2}^{m+n+2}$, and $|p|^2+|q|^2=1$, then we may consider $p+q\epsilon S^{m+n+1}$, $p'=p/|p|\epsilon S^m$, $q'=q/|q|\epsilon S^n$ (unless p or $q=0$). Then

$$\phi_f(p+q)=2|p|\,|q|f(p', q')+(|q|^2-|p|^2)e_{m+n+2}.$$

Using this, a homomorphism $H_{m,n}$ of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$ is defined (R_n=group of rotations of S^n): If $f(p)$ maps S^m into R_n and $q\epsilon S^n$, then $f^*(p, q)=[f(p)](q)$ maps $S^m\times S^n$ into S^n, and ϕ_{f^*} maps S^{m+n+1} into S^{n+1}. L. Pontrjagin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **19**, 147–149 (1938); ibid., 361–363], using a related homomorphism, found $\pi_{n+i}(S^n)$ for $i=1, 2$.

Some of the main results are: $H_{m,n}$ is an isomorphism for $m=1$ and for $m=2$, $n=1$. Let P_n=complex projective n-space, and let θ be the natural mapping of S^{2n+1} onto P_n. Then a mapping f of P_n into a space X can be extended over P_{n+1} if and only if $f\theta$ is inessential. A mapping h' of P_n into S^{2n} is determined, so that it can be extended over P_{n+1} if and only if n is even. This gives a counter-example to a statement of Freudenthal [Nederl. Akad. Wetensch., Proc. **42**, 139–140 (1939)] on the existence of extensions of mappings from the $(q+1)$- to the $(2q-1)$-dimensional part of a complex. *H. Whitney* (Cambridge, Mass.).

Eilenberg, Samuel and MacLane, Saunders. Group extensions and homology. Ann. of Math. (2) **43**, 757–831 (1942). [MF 7406]

A preliminary account of the main results in this paper has already been reviewed [Proc. Nat. Acad. Sci. U.S.A. **27**, 535–539 (1941); these Rev. **3**, 142]. In addition to the details of proof, the present paper contains enough expository material to insure a high degree of completeness and clarity. All groups being Abelian, let $E=\text{Ext }\{G, H\}$ be the totality of classes of equivalent extensions of G with preassigned factor group H, converted into a group by means of a certain composition law; E has a natural topology if G is topological and E is compact if G is. A fundamental result in the purely group-theoretic part of the paper is the interesting characterization of E by means of homomorphisms. Let $\text{Hom }\{R, G\}$ be the group of homomorphic mappings $R\rightarrow G$ and let $\text{Hom }\{F|R, G\}$, R a subgroup of F, be those mappings $R\rightarrow G$ which can be extended to mappings $F\rightarrow G$. If the discrete group H is expressed (as it always can be) in the form F/R, where F is a free group, then

$$\text{Ext }\{G, H\}\cong\text{Hom }\{R, G\}/\text{Hom }\{F|R, G\}.$$

The isomorphism is bicontinuous if G is topological. Among the corollaries to this result is the theorem that a group G possesses extensions other than the trivial (direct product) extensions if and only if every element of G possesses an mth root, $m=2, 3, \cdots$. For compact G and discrete H it is shown that $\text{Ext }\{G, H\}$ is isomorphic to the character group of $\text{Hom }\{G, H\}$.

Turning now to applications to homology theory, let K be a star-finite complex, G a topological group. Denote by $H^q(K, G)$ the qth homology group of K over G (infinite chains and cycles allowed) and by $\mathfrak{H}_q(K, G)$ the qth cohomology group of finite cocycles modulo the coboundaries of finite chains. The main results of the paper are typified by the following theorem: the group $H^q(K, G)$ is expressible in terms of the groups $\mathfrak{H}_q=\mathfrak{H}_q(K, I)$ and $\mathfrak{H}_{q+1}=\mathfrak{H}_{q+1}(K, I)$, where I is the group of integers. An exact description of this relation is the following: Let G and I be paired to G by the product $\phi(m, g)=mg(m\epsilon I, g\epsilon G)$. Then let $H^q(K, G)$ and $\mathfrak{H}_q$ be paired by means of the Kronecker index. Let $Q^q(K, G)$ be the annihilator of $\mathfrak{H}_q$ in $H^q(K, G)$. Then

$Q^q(K, G)$ is a direct factor of $H^q(K, G)$ and is isomorphic in a natural way to Ext $\{G, \mathfrak{H}_{q+1}\}$. Moreover

$$H^q(K, G)/Q^q(K, G)$$

is isomorphic in a natural way to Hom $\{\mathfrak{H}_q, G\}$. These relations yield the decomposition

$$H^q(K, G) \cong \text{Hom } \{\mathfrak{H}_q, G\} \times \text{Ext } \{G, \mathfrak{H}_{q+1}\}.$$

These and related results lead at once to a universal coefficient group for star-finite complexes K: all homology and cohomology groups of K are determined by the groups $\mathfrak{H}_q(K, I)$. By passing to the limits of direct and inverse systems, it is shown that the Čech homology group $\mathfrak{H}^q(X, G)$ of a space X over compact G admits a subgroup $\mathfrak{Q}^q$ such that $\mathfrak{Q}^q$ and $\mathfrak{H}^q/\mathfrak{Q}^q$ are expressible in terms of the integral cohomology groups of X. As for $\mathfrak{H}^q$ itself, results are lacking because it is not known whether $\mathfrak{Q}^q$ is a direct factor of $\mathfrak{H}^q$. In the case of compact metric spaces, however, results are complete. In fact the preceding theorems about complexes can be applied to a star-finite "fundamental complex" associated with a given compact metric space X. It turns out that $\mathfrak{H}^q(X, G)$ is now expressible in terms of $\mathfrak{H}_q(X, I)$ and $\mathfrak{T}_{q+1}(X, I)$, $\mathfrak{T}_{q+1}$ consisting of those elements of $\mathfrak{H}_{q+1}$ which are of finite order. *P. A. Smith.*

Whyburn, G. T. What is a curve? Amer. Math. Monthly **49**, 493–497 (1942). [MF 7340]

This is an expository article pointing out the objections to some classic definitions of curve based on intuitive notions and discussing the modern definition based on the Brouwer-Urysohn-Menger dimension theory. Arcs, simple closed curves, graphs, dendrites and boundary curves are discussed as special types of curves. *W. L. Ayres.*

NUMERICAL AND GRAPHICAL METHODS

***Uhler, Horace S. Original Tables to 137 Decimal Places of Natural Logarithms for Factors of the Form $1\pm n\cdot 10^{-p}$, Enhanced by Auxiliary Tables of Logarithms of Small Integers.** Published by the author, New Haven, Conn., 1942. 120 pp.

These are radix tables similar to, but larger in extent than, those of Flower and others. Table I gives the logarithms of the integers from 2 to 10 and of powers of 10 of the form 10^{10^p} for $p=2, 3, \cdots, 11$. Table II gives the logarithms of numbers of the form $1-n\cdot 10^{-p}$ for $n=1, 2, \cdots, 9$; $p=1, 2, \cdots, 69$. To find the logarithm of a number N, the number is factored into factors of these forms by means of a simple algorithm. To find an antilogarithm, the procedure is reversed by first finding the factored number and then expanding the factors. Table III contains $\log(1+n\cdot 10^{-p})$ for $n=1, \cdots, 9$; $p=1, \cdots, 21$, and can be used alternatively to Table II. Table IV gives logarithms of primes from 11 to 113; also $\log_e 10$ and $\log_{10} e$ to 325 places. Tables V and VI are the same as Tables II and III but in abbreviated notation, and for general p.

P. W. Ketchum (Urbana, Ill.).

***Peters, J. Seven-Place Values of Trigonometric Functions for Every Thousandth of a Degree.** D. Van Nostrand Co., Inc., New York, 1942. $7.50.

A photographic reproduction of tables published in Berlin in 1918. It is pointed out that the tabular error sometimes amounts to .85 units of the last digit.

***Table of** arc tan x. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York as a Report of Official Project No. 165-2-97-22; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1942. xxv+169 pp. $2.00.

The tables give arc tan x (in radians) to 12 decimal places over the following range: 0 (.001) 7 (.01) 50 (.1) 300 (1) 2000 (10) 10000 (figures in parentheses indicate the tabular interval for x between the adjacent limits). The spacing is so chosen that interpolation to 12 decimals may be performed using only second central differences; these are tabulated [an innovation in the W.P.A. tables which greatly increases the usefulness of the tables]. The book includes auxiliary tables to 6 decimal places of $x(1-x)$ and $x(1-x^2)/6$ over the ranges 0(.001).5 and 0(.001)1, respectively; also tables for the conversion of radians to degrees, and vice versa, are given. *W. Feller* (Providence, R. I.).

***Table of Sine and Cosine Integrals for Arguments from 10 to 100.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 265-2-97-11; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1942. xxxii+185 pp. $2.00.

This volume tabulates the values of the sine and cosine integrals

$$\mathrm{Si}(x)=\int_0^x \frac{\sin u}{u}du, \quad \mathrm{Ci}(x)=\int_\infty^x \frac{\cos u}{u}du$$

to 10 decimal places for x between 10 and 100 at intervals of 0.01, thus enormously increasing the usefulness of these important functions. Second central differences are given throughout to aid interpolation by Everett's formula, and a table of interpolation coefficients is provided. The introduction describes the uses of the functions, the method of computing the table, the method of interpolation and supplies a bibliography.

This table is an extension of the Tables of Sine, Cosine and Exponential Integrals already published by the Mathematical Tables Project. Volume I [these Rev. **2**, 239] covers the range from 0 to 2 at intervals of 0.0001, volume II [these Rev. **2**, 366] the range from 0 to 10 at intervals of 0.001. The sine and cosine integrals have a growing practical application not only in theoretical physics but also in such engineering problems as radio-frequency transmission, diffraction, diffusion and net-work design. The three volumes of tables now published put these functions in the class of known functions, available for actual calculation, just like sines, cosines and logarithms.

W. E. Milne (Corvallis, Ore.).

***Table of 5-Point Lagrangean Interpolation Coefficients. (From 0 to 2, Argument 0.001, 7-place.)** Marchant Calculating Machine Company, Oakland, Calif., 1942. 25 pp.

An advance copy of a portion of a more extensive set of tables now in preparation by the Mathematical Tables Project, Work Projects Administration for the City of New York, sponsored by the National Bureau of Standards.

Lowan, Arnold N., Davids, Norman and Levenson, Arthur. Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula Bull. Amer. Math. Soc. **48**, 739–743 (1942). [MF 7276]

This note gives the abscissas $\{x_{i,n}\}$ and the weight-coefficients $\{a_{i,n}\}$ for Gauss' mechanical quadrature formula

$$(1)\quad \int_p^q f(x)dx=((q-p)/2)\sum_{i=1}^{n} a_{i,n} \times f(x_{i,n}(q-p)/2+(q+p)/2)+\rho_n(f),$$

for $n=2, 3, \cdots, 16$. The computation was carried out to 15 decimal places. The expression for the remainder (Markoff) is given, and an illustrative application is made to the integral $\int_1^2 dx/x$. This yields 14 correct decimals for log 2, taking in (1) $n=10$. *J. Shohat.*

Coulson, C. A. and Duncanson, W. E. Some new values for the exponential integral. Philos. Mag. (7) **33**, 754–761 (1942). [MF 7480]

The W.P.A. tables, Tables of Sine, Cosine and Exponential Integrals, Vols. I and II [these Rev. **2**, 239, 366], give the values of the exponential integral

$$\mathrm{Ei}(x)=\int_{-\infty}^{x}\frac{e^t}{t}dt$$

and of $-\mathrm{Ei}(-x)$ for the range $x=0$ to $x=15$. In this paper the authors extend the table from $x=15$ to $x=50$ at unit intervals. The values are given to ten significant figures. The computation was performed by a step-by-step process using a series expansion for the difference $\mathrm{Ei}(x+h)-\mathrm{Ei}(x)$ which is rapidly convergent for small values of h. It is unnecessary to carry the table beyond $x=50$ as the asymptotic expansion of $\mathrm{Ei}(x)$ is satisfactory for $x>50$. The paper contains formulae and supplementary tables to facilitate interpolation. *W. E. Milne* (Corvallis, Ore.).

Krafft, Maximilian. Über ein Eulersches Verfahren zur Wurzelberechnung. Monatsh. Math. Phys. **49**, 312–315 (1941). [MF 6853]

In 1939 [Monatsh. Math. Phys. **48**, 190–197 (1939); these Rev. **1**, 126], W. Lorey discussed an apparently neglected theorem of Euler [Opera Omnia, Ser. I, vol. 6, pp. 240–262]: "Es sei $m>1$ eine natürliche Zahl und r positiv reell. Bildet man ausgehend von beliebigen nichtnegativen Anfangswerten $a_0, b_0, \cdots, k_0, l_0$, die aber nicht alle gleich Null sein dürfen, m Zahlenfolgen $a_\nu, b_\nu, \cdots, k_\nu, l_\nu$ $(\nu=0, 1, 2, \cdots)$ nach folgender Regel: $a_\nu=a_{\nu-1}+b_{\nu-1}+\cdots+k_{\nu-1}+l_{\nu-1}$, $b_\nu=a_\nu+(r-1)a_{\nu-1}$, $c_\nu=b_\nu+(r-1)b_{\nu-1}$, $\cdots$, $l_\nu=k_\nu+(r-1)k_{\nu-1}$, so streben die Quotienten $b_\nu/a_\nu, c_\nu/b_\nu, \cdots, l_\nu/k_\nu$ mit wachsendem ν sämtlich gegen die reelle positive Wurzel $r^{1/m}$." In this form the following should be noted. (1) m preceding "Zahlenfolgen" should be omitted. (2) For better understanding of the theorem the antiquated notation $a_0, b_0, \cdots, k_0, l_0$ should be replaced by $a_{01}, a_{02}, \cdots, a_{0\,m-1}$. (3) The index of the radical is in error; for $r^{1/m}$ read $r^{1/(m-1)}$. Lorey proved the theorem for $m=3,4$. Krafft proves the general theorem, by the use of a related difference equation

$$ry_\nu=y_\nu+\binom{m}{1}(r-1)y_{\nu-1}+\binom{m}{2}(r-1)^2y_{\nu-2}+\cdots+\binom{m}{m}(r-1)^m y_{\nu-m}.$$

A. J. Kempner (Boulder, Colo.).

Higgins, T. J. Note on Whittaker's method for the roots of a power series. Amer. Math. Monthly **49**, 462–465 (1942). [MF 7151]

The author stresses the ease with which Whittaker's explicit formula for the smallest zero of the function $f(z)=-1+a_1z+a_2z^2+\cdots$ may be used to solve certain numerical equations. In an editorial note there is proved the related formula that, if r is a simple zero and if $f(z)$ is analytic and otherwise nonvanishing for $|z|\leqq|r|$, then $r=\lim A_n/A_{n+1}$, where $A_0=1$, $A_n=\sum_1^n a_i A_{n-i}$.

P. W. Ketchum (Urbana, Ill.).

Lowan, A. N., Blanch, G. and Horenstein, W. On the inversion of the q-series associated with Jacobian elliptic functions. Bull. Amer. Math. Soc. **48**, 737–738 (1942). [MF 7275]

This paper deals with the inversion of the series

$$\epsilon=\tfrac{1}{2}\theta_2(0, q^4)/\theta_3(0, q^4)=\sum_{i=0}^{\infty} q^{(2k+1)^2}\Big/\Big(1+2\sum_{k=1}^{\infty} q^{4k^2}\Big).$$

The resulting series for q was first obtained by Weierstrass, who gave the first four terms [Mathematische Werke, vol. 2, Mayer and Müller, Berlin, 1895, p. 276]. Milne-Thompson extended this result to six terms [J. London Math. Soc. **5**, 148–149 (1930), in particular, p. 148]. The present paper gives the first fourteen terms of the series. This result is of importance in the computation of the values of the Jacobian elliptic functions through the use of the theta functions.

M. A. Basoco (Lincoln, Neb.).

Morris, J. Frequency equations. Aircraft Engrg. **14**, 108–110 (1942). [MF 6599]

The author describes a method for the evaluation of the characteristic roots of a finite matrix. The greatest root is found by an iterative process. Then one variable is eliminated thus reducing the order of the matrix. The process is cumbersome. The author seems unaware of the powerful existing methods [for the case of symmetric matrices, see Hotelling, Psychometrika **1**, 27–35 (1936); for the general case and many refinements, see Aitken, Proc. Roy. Soc. Edinburgh. Sect. A. **57**, 269–304 (1937)]. *W. Feller.*

Lonseth, A. T. Systems of linear equations with coefficients subject to error. Ann. Math. Statistics **13**, 332–337 (1942). [MF 7243]

In n simultaneous linear equations in n unknowns, the $n(n+1)$ coefficients are considered to be liable to errors. The error of the ith unknown is expanded as a Taylor's series in the errors of the coefficients, and the terms of the series are evaluated from matrices involving the coefficients, the errors of the coefficients and the true values of the unknowns. If the errors of all the n^2 coefficients of the unknowns are numerically smaller than the absolute value of the determinant of those coefficients divided by the sum of the absolute values of its cofactors, then the errors of the coefficients cannot cause the determinant to vanish. The variance of an unknown is expresed approximately in terms of the variance of the $n(n+1)$ coefficients, in agreement with Etherington's earlier results [Proc. Edinburgh Math. Soc. (2) **3**, 107–117 (1932), in particular, p. 107]. The errors of the coefficients of the unknowns cannot be normally distributed, but must be suitably bounded, if the resulting errors of the unknowns are to remain finite.

T. E. Sterne (Aberdeen, Md.).

Banachiewicz, T. An outline of the Cracovian algorithm of the method of least squares. Astr. J. **50**, 38–41 (1942). [MF 7078]

The author defines expressions called Cracovians, akin to matrices, the only difference being that the product of

Cracovians a times b is defined as a transpose times b for ordinary matrices. Properties of Cracovians are established which are of course merely modifications of corresponding matrix theory. To solve normal equations arising in the method of least squares the procedure (described with matrices instead of the author's Cracovians) is substantially as follows. From the augmented n-row $(n+1)$-column matrix of the normal equations we construct a symmetric square $(n+1)$-rowed matrix R by suitable bordering. We obtain the triangular matrix r (with zeros below the principal diagonal) such that r transpose times r equals R. The lower right hand element of r is taken as -1, which determines the heretofore undetermined lower right hand element of R. Now we obtain the triangular matrix q (with zeros above the principal diagonal) such that r transpose times q equals the unit matrix. The last row of q supplies the unknown. Various checks on the calculation are given. The reviewer is not convinced that the introduction of the term "Cracovian" is desirable, or that the computational procedure is any improvement over methods already known.
W. E. Milne (Corvallis, Ore.).

Spoerl, Charles A. The Whittaker-Henderson graduation formula A. The mixed difference case. Trans. Actuar. Soc. America **43**, 68–80 (1942). [MF 6941]

Spoerl's paper with the above title appeared in the same Trans. **42**, 292–313 (1941) [these Rev. **3**, 155]. The present installment contains the written discussion customary in the Acturial Society; the contributions are by C. M. Sternhell and H. C. Dunkley. *W. Feller* (Providence, R. I.).

Mikeladze, Sch. Über dividierte Differenzen mit wiederholten Argumentwerten. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **9**, 49–60 (1941). (Russian. German summary) [MF 7378]

A formula is established, by mathematical induction, for the divided difference $f(x, x, \cdots, x, a_1, a_2, \cdots, a_n)$, with m coinciding arguments, expressing it in terms of successive derivatives of $f(x)$. This leads, conversely, to expressions with remainder terms for $f'(x)$, $f''(x)$, $\cdots$ in terms of the functional values of $f(x)$ at certain specified points. Finally, the above results are applied to the computation of the characteristic numbers of certain boundary problems (where the corresponding differential equation is reduced first to an equation in finite differences, then to a system of linear equations). *J. A. Shohat* (Philadelphia, Pa.).

Lapauri, I. D. On numerical integration of differential equations of hyperbolic type. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **10**, 93–109 (1941). (Georgian. Russian summary) [MF 7386]

Bertram, Sidney. Calculation of axially symmetric fields. J. Appl. Phys. **13**, 496–502 (1942).

Four place tables of the series

$$\sum_m e^{-mz} J_0(mr)J_1(m)/m^2, \qquad J_0(m)=0,$$

for $m=1, 3$; $r=0(.1).9$; $z=0(.05)1.75$ are given. The characteristics of the functions are discussed and applications to potential distribution over a cylinder of constant radius are indicated. The potential distribution of a simple electron lens consisting of two coaxial cylindrical electrodes of equal diameters is expressed in terms of the tabulated functions. A comparison is made with results found by means of an electrolytic plotting tank. *H. Bateman.*

Pfriem, H. Differenzenverfahren zur Berechnung zeitveränderlicher kugelsymmetrischer Temperaturfelder. Luftfahrtforschung **19**, 197–198 (1942). [MF 7438]

The author makes the [obvious] remark that the finite difference method can be applied to the solution of the heat equation in three dimensions with spherical symmetry.
W. Feller (Providence, R. I.).

Patterson, A. L. and Tunell, George. A method for the summation of the Fourier series used in the x-ray analysis of crystal structures. Amer. Mineralogist **27**, 655–679 (1942). [MF 7464]

A scheme is described for the rapid summation of Fourier series at a set of points dividing the period into N equal intervals. A file of cardboard strips is prepared, each strip bearing values of $D\cos(2\pi k/N)$, $k=0, 1, \cdots, N$, a different strip for each different value of D in the range of coefficients to be considered; and similarly for the sine terms. Strips with values of D equal to the given coefficients are selected from this file and spread out in order in a specially made rack, so as to form an array of all the values of $D\cos(2\pi k/N)$ in the particular series considered. A stencil is then placed over the rack, exposing just those numbers which must be added to give the desired sum. A different stencil is required for each of the points of subdivision of the period. *P. W. Ketchum.*

Macewan, Douglas and Beevers, C. A. A machine for the rapid summation of Fourier series. J. Sci. Instruments **19**, 150–156 (1942). [MF 7409]

This is a mechanization of the Beevers and Lipson "strip" method for the calculation of Fourier syntheses. Extensive use is made of standard relays and rotary switches of types developed for machine switching in telephone systems. The basis of operation is to generate, and distribute to counters, groups of electrical impulses corresponding to the whole-number ordinates which are combined in the "strip" method. Unusually complete schematic details of the system are given. *S. H. Caldwell* (Cambridge, Mass.).

Evans, R. C. and Peiser, H. S. A machine for the computation of structure factors. Proc. Phys. Soc. **54**, 457–462 (1942). (2 plates) [MF 7189]

The authors describe in some detail the construction of a machine to compute $2f\cos(hx-ky)$, where h and k are integers. The accuracy required is low; the maximum error shown in an illustrative table is about $f/20$, and the median error about $f/75$. The mechanism consists of a pair of disk-and-roller multipliers to calculate hx and ky, a differential to add these, and a simple scotch cross-head to produce sinusoidal motion. Much of the mechanism is built of "standard meccano parts." *G. R. Stibitz.*

Brown, S. Leroy and Wheeler, Lisle L. Use of the mechanical multiharmonograph for graphing types of functions and for solution of pairs of non-linear simultaneous equations. Rev. Sci. Instruments **13**, 493–495 (1942). [MF 7460]

Mechanical means are described whereby types of equations may be graphically represented and nonlinear

simultaneous equations may be solved. The procedure requires, in each case, that the equation be changed to a trigonometric form by polar transformation. Graphs are shown to illustrate the processes involved.

R. L. Dietzold (New York, N. Y.).

Knudsen, Lila F. A punched card technique to obtain coefficients of orthogonal polynomials. J. Amer. Statist. Assoc. **37**, 496–506 (1942). [MF 7537]

Katterbach, K. Messen der Krümmung flacher Kurven. Z. Verein. Deutsch. Ingenieure **85**, 449–450 (1941). [MF 6959]

An instrument for measuring the curvature of a plane curve is described. The curve is reproduced by a flexible strip held in a frame and constrained by a number of adjustable parallel rods also held in the frame. One side of the flexible strip is a mirror. The curvature of this curved mirror is measured optically in a device containing a cylindrical lens.

P. W. Ketchum (Urbana, Illinois).

MATHEMATICAL PHYSICS

Silberstein, Ludwik. A fundamental criterion of uniform representability of equiluminous colors on a geometrical surface. J. Opt. Soc. Amer. **32**, 552–556 (1942). [MF 7143]

The author considers the set of all colors C of equal brightness, each defined by two independent stimuli-coordinates u, v (that is, by the quantities of two different monochromatic lights of known wave length) and discusses the possibility of representing $C(u, v)$ as points of a surface S of constant curvature in such a manner that the geodetic distance of two points of S be proportional to the chromaticity difference of the corresponding colors. *I. Opatowski.*

Spencer, Domina Eberle. Illumination from arrays of rectangular sources. J. Opt. Soc. Amer. **32**, 539–551 (1942). [MF 7142]

Exact formulae are developed for the luminous flux density produced at a point P on a given plane surface by arrays of rectangular sources of light distributed in a regular manner over a quadrant of a plane at distance h from P. If the distances between these sources are less than h, certain approximate formulae which the author develops give results correct to within one per cent or better for all situations encountered in engineering practice.

J. S. Frame (Meadville, Pa.).

Frank, Philipp. The influence of an "uneven" anisotropy on the path of light rays. Phys. Rev. (2) **62**, 241–243 (1942). [MF 7196]

General vector relations are derived for the paths of light rays in media of both general local variation and general anisotropy of refractive index, strictly written the realm of geometrical optics (Fermat's principle). The refractive index is then composed of a fundamental isotropic term plus an anisotropic one, both general functions of the vector loci, and the curvature vector of a bundle of deflected rays relative to the corresponding rays of the isotropic case is found in terms of the direction vector and the anisotropic part of the refractive index. More detailed consideration is then given to the simplest case where the latter is equal to the direction vector times a function of the vector loci. Although this case does not apply to ordinary crystal optics, as the anisotropy involved is not even, it covers the following items: the general isotropic medium in a general state of (relative) motion, influence of wind upon the quickest path of an airplane and motion of electrons in space-charge-free electromagnetic fields. *H. G. Baerwald.*

Ambarzumian, V. A. A new method for the calculus of the light dispersion in a turbid medium. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 97–103 (1942). (Russian) [MF 7564]

Pipes, Louis A. Analysis of longitudinal motions of trains by electrical analog. J. Appl. Phys. **13**, 780–786 (1942).

By using the electromechanical correspondence of the current-velocity variety valid for one dimensional systems, a railway train consisting of locomotive and n equal cars is replaced by a uniform low-pass ladder network of n meshes, with series inductance corresponding to the car mass and series combination of capacitance and resistance as shunt impedance corresponding to stiffness and damping of the coupling, the locomotive representing the generator of voltage (force) with internal inductance (mass). The analysis of this electrical system being well known, there are given the general equations in symbolic form, the natural frequencies of the system with and without consideration of damping, the wave analysis (phase velocity) and the response to a suddenly applied constant tracting force. The treatment, with the exception of the general equations and the wave velocity, is restricted to the case: locomotive mass = car mass. Also, in the transient case, damping is omitted.

H. G. Baerwald (Cleveland, Ohio).

Pipes, Louis A. The operational theory of longitudinal impact. J. Appl. Phys. **13**, 503–511 (1942).

The well-known correspondence between the propagation of homogeneous waves in isotropic elastic bars and that of electromagnetic waves along transmission lines is pointed out. The treatment of longitudinal impact for general loads on the far end is carried out via the corresponding transmission line proposition under application of operational calculus. Only the case of constant cross-section (constant line parameters) is considered. The results appear in simpler and more readily applicable form than in the classical theory of Saint-Venant and Boussinesq. *H. G. Baerwald.*

Schelkunoff, S. A. and Feldman, C. B. On radiation from antennas. Proc. I. R. E. **30**, 511–516 (1942). [MF 7462]

This paper deals with the possibilities and limitations of the application of simple transmission-line theory to antennas. It is emphasized that voltage, current and charge are affected by radiation in different ways and that in this context charge and current are preferably considered as primary variables from both the experimental and the theoretical angle, voltage being defined as local charge divided by local capacity. It is shown experimentally and theoretically that, for antennas of lengths $l=n\lambda/2$ fed at current antinodes, the effect of radiation in the charge-current equations can be approximately represented by addition of the term $60l/(l^2-x^2)$ as radiation resistance.

H. G. Baerwald (Cleveland, Ohio).

Kofink, W. und Menzer, E. Reflexion elektromagnetischer Wellen an einer inhomogenen Schicht nach der Wentzel-Kramers-Brillouin-Methode. Ann. Physik (5) 39, 388–402 (1941). [MF 6851]

This paper treats the one-dimensional problem of plane electromagnetic waves in a medium of constant index of refraction up to a certain interface, then a layer of index varying continuously up to a second interface, and then of index having a second constant value. This study is related to the work of R. Gans [Ann. Physik (4) 47, 709–736 (1915)]. The method employed in the present paper is based upon that used in certain analogous problems of wave mechanics. *R. M. Foster* (New York, N. Y.).

Spence, R. D. and Wells, C. P. The propagation of electromagnetic waves in parabolic pipes. Phys. Rev. (2) 62, 58–62 (1942). [MF 7020]

Of the four systems of cylindrical coordinates in which $(\nabla^2+k^2)\Phi=0$ is separable, the parabolic is the only one for which the transmission of electromagnetic waves in pipes has not been treated previously. The pipes considered here are made up of two equal coaxial parabolic segments facing each other. The general solution appears, correspondingly, in terms of confluent hypergeometric (Weber) functions. As with other cross-sectional forms, there are two prototypes, namely, E-waves with $H_z\equiv0$ and H-waves with $E_z\equiv0$, each consisting of even and odd solutions with doubly infinite sequences of nodal configurations characterized by double indices (m, n), where $m=0, 1, \cdots$ designates the order of the Weber functions involved or the number of "azimuthal" nodes and $n=1, 2, \cdots$ the order of the root satisfying the boundary condition, or the number of internal "radial" nodes minus one. The last statement refers to perfectly conducting pipes. For this case and $m=0$, where the Weber functions reduce to Bessel functions of orders $\pm\frac{1}{4}$, the solution is carried out numerically, and the critical wavelengths are given for $n=1$ and $n=2$ for both even and odd E- and H-waves.

For imperfectly, but well, conducting pipes, where the E- and H-types are slightly coupled along the boundary, with predominance of one type, the field inside the metal is adequately described by the principal asymptotic terms of the Weber functions. The attenuation constants are the quantities of chief interest. They are obtained via the Poynting vector by application of the "associated impedance" concept, in the form of integrals from which the dependence on frequency is apparent. This is the same ($\sim f^{\frac{1}{2}}$ for E-waves, with a term $\sim f^{-\frac{3}{2}}$ for H-waves) as for other cylinders. Thus the general characteristics of transmission are the same for all pipes with cross-sections corresponding to separable cases of the wave equation. *H. G. Baerwald* (Cleveland, Ohio).

Weber, Ernst. Traveling waves on transmission lines. Elec. Engrg. 61, 302–309 (1942).

An expository article on the applications of the Laplace transform to the solution of the transmission line equations. *A. E. Heins* (Cambridge, Mass.).

Hsü, Chang-Pen. Transmission theory of a cylindrical hollow tube guide. J. Math. Phys. Mass. Inst. Tech. 21, 23–42 (1942). [MF 6777]

Hsü, Chang-Pen. Transmission theory of concentric lines. J. Math. Phys. Mass. Inst. Tech. 21, 43–51 (1942). [MF 6778]

In these two papers the author studies the merits of the transmission systems (a) a concentric transmission line (with current flowing down an insulated center cable and back along a thick outer cable), and (b) a linear antenna located on the axis of a hollow metallic tube guide. The computations are only sketched and depend on the Sommerfeld-Weyrich approach to solutions of Maxwell's equations. The main comparative conclusions are: (1) In both systems (a) and (b) it is possible to match a transmitter to the transmission system so that all energy is completely transmitted or radiated (=perfect match). For (b), however, no perfectly matching receiver is conceivable at the present stage of knowledge. (2) Reflection and standing waves: none in (a); unavoidable in (b), leading to interference between receiver and transmitter, and distortion of signals. Other comparative results are listed, most of which are unfavorable to (b) for long distance transmission. *A. L. Foster* (Berkeley, Calif.).

Sullivan, W. L. Analysis of systems with known transmission-frequency characteristics by Fourier integrals. Elec. Engrg. 61, 248–256 (1942).

This is the fourth of a group of five lectures given before the American Institute of Electrical Engineers as a symposium on "Advanced Mathematics as Applied to Electrical Engineering." Consider any electrical network that responds to each applied sinusoidal electromotive force with a sinusoidal output electromotive force having the same frequency. Also let the network be linear so that the principle of superposition applies. The transmission-frequency characteristics are the two functions of the frequency that give the ratio of the amplitudes of the applied and output forces, and the phase difference of those forces. Given these two characteristics the transient output of the network corresponding to an arbitrary prescribed applied force can be determined by modifying the Fourier integral of the applied force. This fundamental application of Fourier integrals is described and illustrated here. *R. V. Churchill.*

Frankel, Sidney. Characteristic impedance of parallel wires in rectangular troughs. Proc. I. R. E. 30, 182–190 (1942). [MF 6909]

The determination of the characteristic impedance reduces to a two-dimensional problem in electrostatics, the logarithmic potential. The solution given in this paper is based on the use of conformal transformations, together with the method of images. It is assumed that the wires are perfect conductors, of circular cross-section, and of diameter small compared to the distance between them and to the distance from the wire to any side of the trough. This general method of solution is applied to two specific problems: the balanced 2-wire transmission line, and the balanced 3-wire transmission line in which the center lines of the three wires lie in a plane. In each case, the wires are surrounded by perfect, grounded, conducting surfaces, symmetrically situated to form a trough. *R. M. Foster.*

Dwight, Herbert B. Formulas for the magnetic-field strength near a cylindrical coil. Elec. Engrg. 61, 327–333 (1942).

This is a collection of expansional formulas for the magnetic field distribution about solenoids. They are grouped into expansions with predominant logarithmic term, representing the field close to the conductor, and those in terms of Legendre polynomials, valid for moderate and large distances; the closed expressions in terms of complete elliptic integrals are also given, but only for the simplest cases as otherwise they are not useful for practical com-

putation. The zonal harmonics expansions are given for short, long and infinitely long coils. Both types of expansions are developed proceeding from the circle via the thin solenoid to the coil of arbitrary thickness. All formulas except those for one turn refer to an end plane of the coil; the field at other axial distances follows as difference of the fields in end planes of two adjacent solenoids. The approximate ranges of practical applicability of the different expansions up to the terms included is represented in a diagram in terms of the coil dimensions and the radial distance; there is ample overlapping. Four numerical examples are included. *H. G. Baerwald.*

Millman, Jacob. Laplacian transform analysis of circuits with linear lumped parameters. Elec. Engrg. **61**, 197–205 (1942).

This is the third of a group of five lectures given before the American Institute of Electrical Engineers as a symposium on "Advanced Mathematics as Applied to Electrical Engineering." It is an exposition of some of the basic properties of the Laplace transformation, with applications to systems of ordinary differential equations in electric circuit theory. The recent book by Carslaw and Jaeger [Operational Methods in Applied Mathematics, Oxford University Press, New York, 1941; these Rev. **3**, 243], which evidently appeared too late for this lecture, should now be mentioned as a useful addition to the author's list of references. *R. V. Churchill* (Ann Arbor, Mich.).

Coulthard, W. B. Operational methods of dealing with circuits excited by sinusoidal impulses. Canadian J. Research. Sect. A. **20**, 33–38 (1942). [MF 6401]

Rectified alternating-current waves and the response of a parallel resistance-capacitance circuit to these are treated formally by a mixture of Cauchy-Heaviside-operational and Laplace-transformation methods. Results are given in infinite series form. The presentation is obscure. Pulses are called impulses. The reciprocal of a function is called its inverse. The introduction contains the incorrect statement that the method is applicable equally to waves of any shape. *J. L. Barnes* (Princeton, N. J.).

Whiteman, R. A. A contribution to the theory of network synthesis. Proc. I. R. E. **30**, 244–246 (1942). [MF 6910]

Impedance functions of electrical networks are written as quotients of Laplace integrals (of voltage and current, etc.). As an application, an equivalent steady state impedance is defined for nonlinear resistances; as is evident, this equivalence depends on the time function (transient) applied, in addition to the resistance characteristic. *H. G. Baerwald* (Cleveland, Ohio).

Waidelich, D. L. Steady state currents of electrical networks. J. Appl. Phys. **13**, 706–712 (1942).

As the practical computation of responses of electrical networks to non-periodic signals by the classical analytical methods [Bromwich-Wagner-integral] is often tedious, engineering methods introducing repetitive signals of slow sequence [approximately the continuous signal spectrum by a line spectrum] are frequently resorted to. The [approximative] results are then obtained as Fourier series with coefficients explicitly determined by the transfer factors of the networks concerned. Occasionally, however, it is useful to proceed in the opposite way; that is, to apply transient methods to the case of periodic signals, in preference of the Fourier series expansion. This is the case if (1) the transient response to a single signal period is easy to obtain and if (2) the wave shape rather than the harmonic content is of predominant interest. In the present paper, this idea is carried out along conventional lines and the steady state is given in three forms; namely, as faltungs-integral with the periodic instead of the transient indicial admittance as kernel, as finite sum of signal Fourier series for the case of finite lumped networks, in correspondence with the Heaviside expansion formula, and finally as contour integral, corresponding to the Bromwich-Wagner integral, with equidistant "signal" poles along the imaginary axis. A simple example is given. *H. G. Baerwald.*

Higgins, Thomas James. The vector potential and inductance of a circuit comprising linear conductors of different permeability. J. Appl. Phys. **13**, 390–398 (1942).

The low-frequency formula for the inductance of a circuit composed of a long circular cylinder and an outer eccentrically located circular cylinder are derived for the case in which the permeabilities of each conductor and the surrounding medium are different. The solution also may be interpreted as that of an analogous problem in heat conduction. *J. L. Barnes* (Princeton, N. J.).

Higgins, Thomas James. Formulas for the inductance of rectangular tubular conductors. J. Appl. Phys. **13**, 712–715 (1942).

This is a supplement to a paper by this author under the same title [Trans. Amer. Inst. Elec. Engrs. **60**, 1046–1050 (1941); these Rev. **3**, 256]. It provides an independent check formula and an example of its use. *J. L. Barnes.*

Lifshitz, Jaime. On the Fourier analysis of orbits in the equatorial plane of a magnetic dipole. J. Math. Phys. Mass. Inst. Tech. **21**, 94–116 (1942). [MF 7250]

In the theory of primary cosmic rays, the rays are considered to be charged particles moving in the earth's magnetic field, which may be regarded as a dipole. The equations of motion have been solved in general only for the case of motion in the equatorial plane of the dipole. The present paper deals with the behavior of orbits in the neighborhood of the equatorial plane; and is concerned in particular with the solution of certain perturbation equations of the form $y''=f(x)y$, where the coefficient $f(x)$ is a doubly periodic function obtained from the equatorial motion. The classical methods of Hill can be used to solve this equation if the Fourier expansion of $f(x)$ is known. These Fourier coefficients are determined for $f(x)$ equal to

$$(a+b\,\mathrm{sn}^2\,kx)/(c+d\,\mathrm{sn}^2\,kx)$$

and also for the logarithm of this same expression. The necessary integrals are evaluated by contour integration around a period rectangle. *P. W. Ketchum.*

Galanin, A. D. Die Bewegung des Mesons im homogenen magnetischen Feld. Acad. Sci. USSR. J. Phys. **6**, 27–34 (1942). [MF 7412]

The problem of the motion of vector mesons in a homogeneous magnetic field is solved by making use of Tamm's [C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 551–555 (1940); these Rev. **3**, 319] form of the meson equations. The characteristic energy values show that the meson possesses a definite magnetic moment at all energies in contrast to the behaviour of a Dirac electron whose moment goes to

zero for motions with relativistic velocities. Expressions for charge, current and momentum densities are also given.
L. W. Nordheim (Durham, N. C.).

Ma, S. T. and Yu, F. C. Electromagnetic properties of nuclei in the meson theory. Phys. Rev. (2) 62, 118–126 (1942). [MF 7094]

Following a procedure of Pauli [Rev. Modern Phys. 13, 203–232 (1941)] general expressions for the electromagnetic current vector in a meson field produced by nuclear particles are derived with the help of the gauge invariance of the total Lagrange function. The method can be applied to meson fields of all types and explicit expressions are given for the vector and for the pseudoscalar case. It permits one to treat fully the interaction between nuclear particles, including their exchange meson field with external electromagnetic fields. The electric and magnetic moments of a neutron-proton pair due to the exchange currents are calculated. It is thus shown that a deuteron in a stationary state has no extra nonadditive magnetic moment due to meson exchange for both the vector and the pseudoscalar theory. In the pseudoscalar theory a deuteron possesses also no exchange dipole or quadripole electrostatic moment in contrast to the vector case.
L. W. Nordheim.

Pauli, W. and Dancoff, S. M. The pseudoscalar meson field with strong coupling. Phys. Rev. (2) 62, 85–108 (1942). [MF 7093]

In this paper a first full report is given on the strong coupling approximation for the treatment of the meson fields as produced by nucleons (that is, protons and neutrons). In this method a finite size of the source of the meson field (that is, the nucleons) is assumed which will be of the order $a=\hbar/(Mc)$, where M is the mass of a nucleon. Then the lowest characteristic values of the interaction energy of meson field and nuclear particle is split off under the assumption that the next higher levels will not appreciably disturb a system in this lowest state. The conditions for the validity of this procedure are a $k=a\mu c/\hbar\gg 1$ (μ=meson mass) and a coupling constant $g^2/\hbar c\gg(ka)^{\frac{1}{2}}$. The cases of a charged and of a symmetrical (with respect to charged and neutral mesons) pseudoscalar meson field are treated, since vector mesons seem to lead to too strong electromagnetic radiation processes to be compatible with cosmic ray evidence. Only the case of a single source, that is, a single nucleon, is considered and the following results are obtained. The theory leads automatically to the existence of higher states with higher spins and charges for the nucleons, the neutron and proton corresponding to the lowest states with spin $\frac{1}{2}$. The scattering of free mesons by nuclei is reduced sufficiently by the presence of the excited states to be compatible with cosmic ray evidence [compare Heitler and Ma, Proc. Roy. Soc. London. Ser. A. 176, 368–397 (1940)]. Finally the magnetic moment of the system nucleon plus meson field is evaluated. The theory would lead to opposite and equal moments for protons and neutrons so that the anomalous moments do not follow in a simple way from this theory.
L. W. Nordheim.

Galanin
Galinin, A. D. Untersuchung der Eigenschaften des Elektronen- und Mesonenspins in der klassischen Näherung. Acad. Sci. USSR. J. Phys. 6, 36–47 (1942). [MF 7413]

The transition to the classical limit ($\hbar\to 0$) is carried out for the equations of motion of vector mesons. In the zeroth approximation the classical Hamilton-Jacobi equation is obtained for the motion of the charge (not the mass as in ordinary mechanics). The next approximation yields the laws for the precession of the spin. The results are compared with a similar development for Dirac electrons and the case of a constant electromagnetic field is investigated in detail. At small velocities both the electron and the meson behave like classical rotating gyroscopes, the Thomas factor for the meson being zero. At relativistic velocities the magnetic moment of the electron goes to zero while the moment of the meson remains intact though its behaviour is rather complicated.
L. W. Nordheim (Durham, N. C.).

Madhava Rao, B. S. Commutation rules related to particles of spins half and one. J. Mysore Univ. Sect. B. 3, 59–63 (1942). [MF 7155]

In a previous paper [Proc. Indian Acad. Sci., Sect. A. 15, 139–147 (1942); these Rev. 4, 31] the author derived, in the case of particles of higher spin, the commutation rules for the matrices β_μ of the wave equation in Dirac form (1) $\partial_\mu\beta_\mu\psi+\chi\psi=0$. In the present paper he seeks to determine an arbitrary numerical constant k which occurred in his commutation rules, by adding to his three original assumptions the hypothesis that the second order equation derivable from (1) by means of his commutation rules has the usual form (2) $\partial_\mu\partial_\mu\psi=\chi^2\psi$. He shows that for particles of spin $\frac{1}{2}$, 0 and 1 this method determines k uniquely, and the resulting commutation rules are the familiar ones for these cases. He states that for particles of spin $\frac{3}{2}$ and 2, however, the method leads to two alternative values of k. Further criteria which will determine k uniquely in these cases are promised in another paper.
O. Frink.

Heitler, W. The influence of radiation damping on the scattering of light and mesons by free particles. I. Proc. Cambridge Philos. Soc. 37, 291–300 (1941). [MF 7102]

The quantum mechanical computation of scattering of photons and mesons is refined by inclusion of the effects of radiation damping. The method consists in the consideration of all accessible intermediate states including those in which two scattered quanta are present. It is possible to solve the problem following the method of Wigner and Weisskopf, if it is assumed that the interaction is switched on adiabatically. An integral equation for the amplitudes of the intermediate states in equilibrium with the amplitude of the initial state is obtained which can be solved approximately for simple cases. The results are very similar to those of classical theory when the Lorentz self force is included. Radiation damping prevents an unlimited increase of all cross sections at high energies, also for the scattering of vector mesons by nuclei, and removes thus part of the difficulties encountered in the usual treatment of high energy processes.
L. W. Nordheim (Durham, N. C.).

Wilson, A. H. The quantum theory of radiation damping. Proc. Cambridge Philos. Soc. 37, 301–316 (1941). [MF 7195]

The same problem as in the companion paper by Heitler [see the preceding review] is treated independently by a somewhat different method. In place of the scattering of light by a free electron, the scattering is first obtained for a bound electron whose binding energy is small compared to the energy of the light quantum. The binding energy is then allowed to go to zero and the scattering by a free particle, initially at rest, is obtained. The results, in substantial agreement with those of Heitler, show that the Klein-Nishina formula for energetic quanta is practically unaf-

fected, but that meson scattering cross sections are very considerably reduced. *L. W. Nordheim.*

Sokolow, A. Die Streuung der Mesonen unter Berücksichtigung der Dämpfung. Acad. Sci. USSR. J. Phys. 5, 231–237 (1941). [MF 6696]

The scattering of mesons by nuclear particles is treated quantum mechanically including the effects of radiative damping, that is, the reaction of the nuclear particles to the meson field. The results are analogous to the independent work of Heitler and Wilson [see the two preceding reviews] on the same subject. The nonrelativistic case is worked out in detail. The quantum mechanical methods lead to the same results as obtained in classical theory and all cross sections show a regular behavior at high energies.

L. W. Nordheim (Durham, N. C.).

March, A. Ganzzahligkeit in Raum und Zeit. IV. Z. Phys. 115, 522–529 (1940). [MF 7322]

This is the fourth of a series of papers dealing with the theory of a micro-physical metric characterized by the length l_0 [see the following review]. It discusses the inequality:

$$|\Delta p|^2 - \frac{1}{c^2}|\Delta E|^2 < (h/2l_0)^2.$$

This relation restricts the change of linear momentum (Δp) and energy (ΔE) for an arbitrary particle. The same relation with a somewhat different physical interpretation was formulated by Heisenberg [Z. Phys. 110, 251–266 (1938)]. The author discusses the difference in these interpretations and the connection between this formula and experiment in the case of showers, creation of heavy electrons by photons, emission and absorption. *L. Infeld.*

March, A. Raum, Zeit und Naturgesetze. Z. Phys. 117, 413–436 (1941). [MF 6845]

This is the author's fifth paper dealing with the theory of measurements [March and Foradori, Z. Phys. 114, 215–226, 653–666 (1939); March, ibid., 115, 245–256, 522–529 (1940); these Rev. 1, 184, 352, and the preceding review]. The theory is characterized by the appearance of l_0, that is, the elementary length, of the order of the radius of an electron. The logical assumptions of this theory are reviewed and deepened. It is stressed that the theory does not change the ordinary concepts of geometry; what it does change is the connection between these concepts and reality. This connection is ordinarily established by the following axiom: if A coincides with B and B coincides with C, then A coincides with C. The author denies the validity of this axiom and it is by this negation that the parameter l_0 is introduced. He deduces from this theory the inequality mentioned in the preceding review. *L. Infeld* (Toronto, Ont.).

Mayer, Joseph E. Contribution to statistical mechanics. J. Chem. Phys. 10, 629–643 (1942). [MF 7263]

This paper deals with very general chemical systems and extends to them many results which are classic in such simple systems as perfect gases. The central point is the calculation of the distribution function $F_n(z, \{\mathbf{n}\})$ as a power series in the fugacity z, defined by

$$z = \lim_{\rho_0 \to 0} \rho_0 \exp\left((\mu - \mu_0)/kT\right),$$

in which ρ_0 is the density, μ, μ_0 being the chemical potentials of the system in general and at density ρ_0, and k, T the Boltzmann constant and absolute temperature. More descriptively, z is the density in particles per unit volume of a fictitious system composed of the same particles but with mutual forces miraculously annihilated, the fictitious system remaining in osmotic equilibrium with the given one. Finally, n is the number of particles of f degrees of freedom composing our system, and $\{\mathbf{n}\}$ symbolizes its nf coordinates. $F_n(0, \{\mathbf{n}\})$, being the distribution of a relatively simple system, can be found readily by replacing certain intermolecular forces by their averages, the other coefficients in the expansion are obtained. The method is extended to the expansion of $F_n(z, \{\mathbf{n}\})$ about any general fugacity $z = z_0$.

Using methods employed by the author in his study of imperfect gases [see also the discussion by Born and Fuchs, Proc. Roy. Soc. London. Ser. A. 166, 391–414 (1938), in particular, p. 391], similar power series are derived for the pressure and density. All these power series represent functions which are regular on the real positive axis of fugacity except at the points characteristic of the phase-transitions of the system, where their study is particularly rewarding.

B. O. Koopman (New York, N. Y.).

Benham, W. E. The nature of temperature. Proc. Phys. Soc. 54, 121–128 (1942). [MF 7106]

The author feels that there is some difficulty at first sight in reconciling temperature as energy per unit mass with the concept of temperature obtainable from radiation theory. The paper is a more or less speculative discussion, adducing evidence to support the view that the ultimate significance of temperature is that it is measured by the thinness of a pulse of electromagnetic radiation. Some preliminary remarks on a new theory of radiation (which requires that central orbits shall be nonradiating when circular) are used to support a dimensional treatment in which the energy density of radiation is a function of the mass (rather than of the charge) of an electron and the absolute temperature. *B. O. Koopman.*

Wheeler, T. S. The energy of the $1s2s\,^3S$ state of the helium atom and related two-electron ions. Proc. Roy. Irish Acad. Sect. A, 48, 43–53 (1942). [MF 7156]

The energy of the lowest triplet state of the helium atom (and of heavier two-electron ions) is computed by a variational method. The author uses the simple variation function

$$\Phi = 1s_{\beta_1 Z}(1)2s_{\beta_2 Z}(2) - 2s_{\beta_2 Z}(1)1s_{\beta_1 Z}(2),$$

where Z is the nuclear charge of the ion, $1s_{\beta_1 Z}$ and $2s_{\beta_2 Z}$ are the $1s$ and $2s$ wave functions of hydrogen-like atoms with nuclear charges $\beta_1 Z$ and $\beta_2 Z$, respectively. The energy integral is minimized with respect to the parameters β_1 and β_2, and the energies sought for are computed for all values of Z between 2 (He) and $8(O^{6+})$. They agree, within reasonable limits, with the available experimental data (for He and Li^+) and with the results of previous more elaborate computations [see Hylleraas, Z. Phys. 54, 347–366 (1929); Hylleraas and Undheim, Z. Phys. 65, 759–772 (1930)]. The author's choice of Φ has the advantage that all the integrals occurring can be evaluated analytically without using graphical methods. *V. Bargmann* (Princeton, N. J.).

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